OPERATIONS RESEARCH

The ICFAI University

Operations Research

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Chapter I

What is Operations Research?

After reading this chapter, you will be conversant with:

- Origin and History
- Areas of Applications
- Phases and Processes of Operations Research
- Assumptions of Operations Research
- Modeling in Operations Research
- General Methods for Solving Operations Research Models
- Role of Computers in Operations Research
- Limitations of Operations Research

Introduction

We make decisions in every day life without even noticing them. In simple situations decisions are taken by common sense, sound judgment and expertise without using any mathematics. But in some situations, the decisions we are concerned with are rather complex and heavily loaded with responsibility. Examples of such decision are, finding the appropriate product mix when there are large numbers of products with different profit contributions and production requirement, planning public transportation network in a town having its own layout of industrial areas, commercial areas, residential areas etc. Certainly, in such situations decisions may well be arrived at intuitively from experience and common sense, yet they are more judicious if backed up by mathematical reasoning. A decision, which takes into account all the present circumstances can be considered the best one, is called optimal decision. The search for a decision may also be done by trial and error but search may be cumbersome and costly. Preparative calculations may avoid long and costly research. Doing preparative calculation is the purpose of Operations Research (OR), which begins when some mathematical and quantitative techniques are used to substantiate the decision being taken. Therefore, Operations Research can be defined as the science of decision-making. Operations Research does mathematical scoring of consequences of a decision with the aim of optimizing the use of time, efforts, resources and avoiding blunders.

Definition

OR is the application of scientific methods to problems arising from operations involving integrated systems of men, machines, money and materials. It normally utilizes the knowledge and skill of an inter-disciplinary research team to provide the managers of such systems with optimum operating solutions. Thus, OR may be defined as a scientific approach for problem-solving to make optimal decisions by executive management.

Some key steps in Operations Research that are needed for effective decision-making are:

- Problem Formulation (motivation, short- and long-term objectives, decision variables, control parameters, constraints).
- Mathematical Modeling (representation of complex systems by analytical or numerical models, relationships between variables, performance metrics).
- Data Collection (model inputs, system observations, validation, tracking of performance metrics).
- Solution Methods.
- Validation and Analysis (model testing, caliberation, sensitivity analysis, model robustness).
- Interpretation and Implementation (solution ranges, trade-offs, visual or graphical representation of results, decision support systems).

These steps require a solid background in mathematics and familiarity with other disciplines (such as physics, economics, and engineering), as well as clear thinking and intuition. The mathematical sciences prepare students to apply tools and techniques and use a logical process to analyze and solve problems.

Operations Research can also be treated as science devoted to describing, understanding and predicting the behavior of systems, particularly man-machine systems. Thus, OR workers are engaged in three classical aspects of science. They are:

- Describing the behavior of systems.
- Analyzing this behavior by constructing appropriate models.
- Using these models to predict future behavior.

It has been successful in providing a systematic and scientific approach to all kinds of government, military, manufacturing, and service operations. Operations Research is a splendid area for graduates of mathematics to use their knowledge and skills in creative ways to solve complex problems and have an impact on critical decisions.

There are now many Operations Research departments in industry, government, and academia throughout the world. Areas where Operations Research has been successful in recent years include:

- Airline Industry (routing and flight plans, crew scheduling, revenue management).
- Telecommunications (network routing, queue control).
- Manufacturing (system throughput and bottleneck analysis, inventory control, production scheduling, capacity planning).
- Healthcare (hospital management, facility design).
- Transportation (traffic control, logistics, network flow, airport terminal layout, location planning).

There are many mathematical techniques that were developed specifically for Operations Research applications. These techniques arose from basic mathematical ideas and became major areas of expertise for industrial operations.

One important area of such techniques is optimization. Many problems in industry require finding the maximum or minimum of an objective function of a set of decision variables, subject to a set of constraints on those variables. Typical objectives are maximum profit, minimum cost, or minimum delay. Frequently there are many decision variables and the solution is not obvious. Techniques of mathematical programming for optimization include: linear programming (optimization where both the objective function and constraints depend linearly on the decision variables), non-linear programming (non-linear objective function or constraints), integer programming (decision variables restricted to integer solutions), stochastic programming (uncertainty in model parameter values), and dynamic programming (stage-wise, nested, and periodic decision-making).

ORIGIN AND HISTORY

Although foundations were laid earlier, the field of operations research as we know it arose during World War II, as military planners in the United Kingdom and in the United States looked for ways to make better decisions in such areas as logistics and training schedules. During this war, a team of scientists of different disciplines came together to solve a number of war related-problems where limitation of resources was an important consideration. It was realized at the end of the war that, although the problems related to war were no longer present, the techniques developed in the process were utilized for civilian problems as well. This was how the subject operations research developed in an integrated manner to deal with the various types of operational problems.

After the war it began to be applied to similar problems in industry. Operations Research emerged as a systematic and integrated subject at the end of World War II.

It is known as “operational research” in the United Kingdom and as “operations research” in most other English-speaking countries, though OR is a common abbreviation everywhere. As the name implies ‘Operations Research’ was apparently invented because the team was dealing with research on (military) operations. The work of this team of scientists was named as Operational Research in England.

One of the earliest applications of analytical methods in operational problem was carried out in early twentieth century by a Danish mathematician, Erlang. The focus of his study was to recommend ways and means of improving the telephone service. Based on the data of demand pattern of the subscribers and the available equipments to manage calls, he used analytical methods which provided insights into the system of telephone usage and subsequently improved subscriber’s service.

India was one of the few countries that started using OR. In 1949, the first OR unit was established in the Regional Research Laboratory at Hyderabad. At about the same time another group was set up in the Defense Science Laboratory to solve the problems of stores, purchase and planning. In 1953, an OR unit was established in the Indian Statistical Institute, Calcutta, with the aim of using OR methods in national planning and survey. OR Society of India was formed in 1955. The society is one of the first members of International Federation of OR societies. The society started publishing OPSEARCH, a learned journal on the subject in 1963. Today OR is a popular subject in management institutes and schools of mathematics and is gaining currency in industrial establishments.

AREAS OF APPLICATION

A few examples of applications in which operations research is currently used include the following:

- Designing the layout of a **factory** for efficient flow of materials.
- Constructing a **telecommunications** network at low cost while still guaranteeing quality service if particular connections become very busy or get damaged.
- Determining the routes of **school buses** so that as few buses are needed as possible.
- Designing the layout of a **computer chip** to reduce manufacturing time (therefore reducing cost).
- Managing the **flow of raw materials** and products in a supply chain based on uncertain demand for the finished products.
- Planning cash flow, analyzing credit policy.
- Scheduling of training programs, personnel planning, recruiting employees.
- Planning dividend policy making, analyzing investment and portfolio.
- Advertising budget location, product selection and introduction timing.
- Planning for materials transfer, optimal buying and optimal reordering.
- Scheduling Research and Development.

PHASES AND PROCESSES OF OPERATIONS RESEARCH

OR is a logical and systematic approach to provide a rational basis for decision-making. The phases and processes of OR study are quite logical and systematic. There are six important steps in OR study, but it is not necessary for each and every step to be invariably present in all studies. These steps are arranged in the following logical order:

Step 1 : Observe the Problem Environment

The activities involved in this step are visits, conferences, observations, research etc. The OR scientist gets sufficient information and support to proceed and is better at formulating the problem, with such activities.

Step 2 : Analyze and Define the Problem

This step not only defines but also stresses uses, objectives and limitations of the problem.

Step 3: Develop a Model

OR models are basically mathematical models representing systems, processes or environment in the form of equations, relationships or formulae. The activities in this step defines interrelationships among variables, formulating equations, using known OR models or searching alternate models. The proposed model may be field tested and modified in order to work under stated environmental constraints. A model may also be modified if the management is not satisfied with the answer it gives.

Step 4 : Select Appropriate Data Input

No model will work appropriately if data input is not appropriate. Hence, tapping right kind of data is a vital step in OR process. Important activities in this step are analyzing internal-external data and facts, collecting opinions and using computer data banks. The aim of this step is to have sufficient input to operate and test the model.

Step 5 : Provide a Solution and Test Reasonableness

This step explains how to get a solution with the help of model and data input. Such a solution is not implemented immediately.

- Use the solution to test the model.
- Find limitations if any.

If the solution is not reasonable, or if the model is not behaving properly, updating and modification of the model are considered at this stage. The end result of this step is the solution that is desirable and supports current organizational objective.

Step 6 : Implement Solution

This is the last step of OR process, where decision-making is scientific but implementation of decision involves many behavioral issues. Therefore, the implementing authority has to resolve the behavioral issues.

ASSUMPTIONS OF OPERATIONS RESEARCH

To solve a problem by using the techniques of OR , the following assumptions are made:

- The available resources and the consumption per unit of the products manufactured from these resources are known with certainty.
- The total units of a constrained resource used equals sum of the individual resource requirements of the products in the given product mix.
- The products can be produced in fractions and resources required in their manufacture too can be employed in fraction of a unit.
- The problem involves only one objective (for example, profit maximization, cost minimization, etc.
- All external factors are stationary and they remain so over the period (i.e., selling price, variable cost and resource requirement per unit of the product remain constant over the entire range of the output).

MODELING IN OPERATIONS RESEARCH

A model is defined as a representation of an actual object or situation. It shows the relationships (direct or indirect) and inter-relationships of action and reaction in terms of cause and effect.

The main objective of a model is to provide means for analyzing the behavior of the system for the purpose of improving its performance.

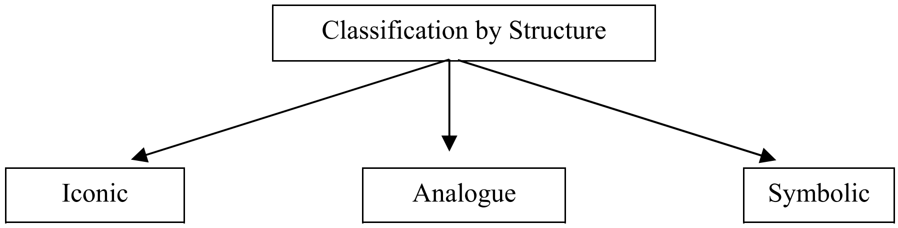
Models can be classified according to the following characteristics:

Classification by,

- Structure
- Purpose
- Nature of environment
- Behavior
- Method of solution
- Use of digital computers.

CLASSIFICATION BY STRUCTURE

Figure 1



Iconic Models: These models represent the system as it is by scaling it up or down (i.e., enlarging or reducing the size). In other words, it is an image.

For example, photographs, drawings, maps etc.

Analogue Models: The models, in which one set of properties is used to represent another set of properties are called analogue models.

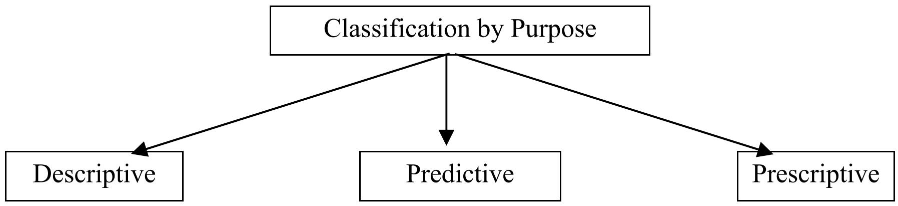
For example, Graphs, Contour-lines on map etc.

Symbolic (Mathematical) Models: It is the model which employs a set of mathematical symbols to represent the decision variables of the system.

For example, letters, numbers etc.

CLASSIFICATION BY PURPOSE

Figure 2



Models can also be classified by purpose of its utility. The purpose of a model may be (a) Descriptive, (b) Predictive, and (c) Prescriptive.

Descriptive Models: This model describes some aspects of a situation based on observations, survey, questionnaire results or other available data.

For example, opinion poll.

Predictive Models: These models can make predictions regarding certain events.

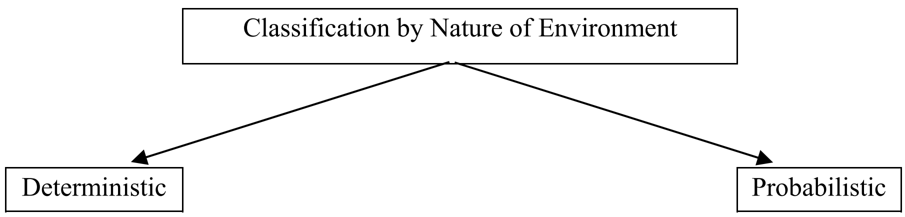
For example, based on the survey results, television networks, such models attempt to explain and predict the election results before all the votes are actually counted.

Prescriptive (Normative) Models: When a predictive model has been successful, it can be used to prescribe a source of action.

For example, Linear programming problem (because it prescribes what managers ought to do).

CLASSIFICATION BY NATURE OF ENVIRONMENT

Figure 3



Deterministic Models: These models assume conditions of complete certainty and perfect knowledge.

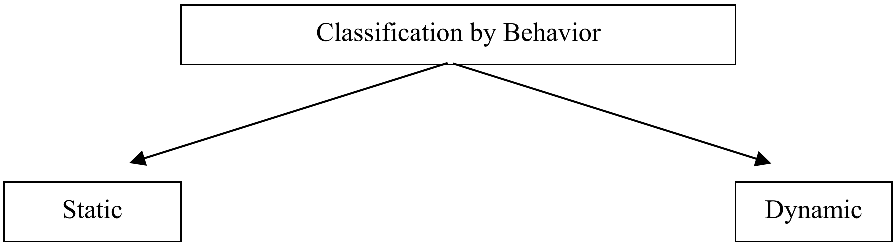
For example, Linear programming, Transportation and Assignment models.

Probabilistic (Stochastic) Models: These models usually handle situations in which consequences or pay-off of managerial actions cannot be predicted with certainty.

For example, risk of fire, accidents, sickness and so on.

CLASSIFICATION BY BEHAVIOR

Figure 4

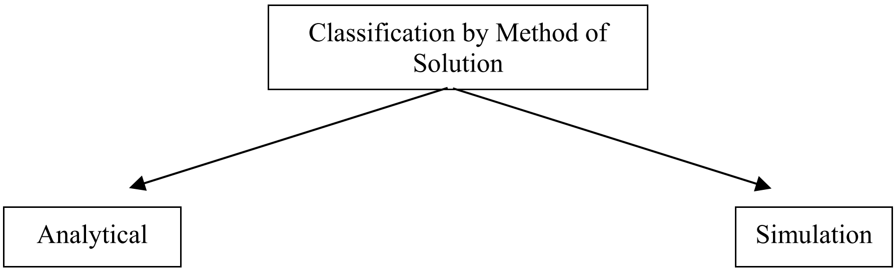


Static Models: These models do not consider the impact of changes that take place during the planning horizon, i.e., they are independent of time. In static model only one decision is needed for duration of a given time period.

Dynamic Models: In these models, time is considered as one of the important variables and it admits impact of changes generated by time. Not only one but also series of interdependent decisions are required during the planning horizon.

CLASSIFICATION BY METHOD OF SOLUTION

Figure 5



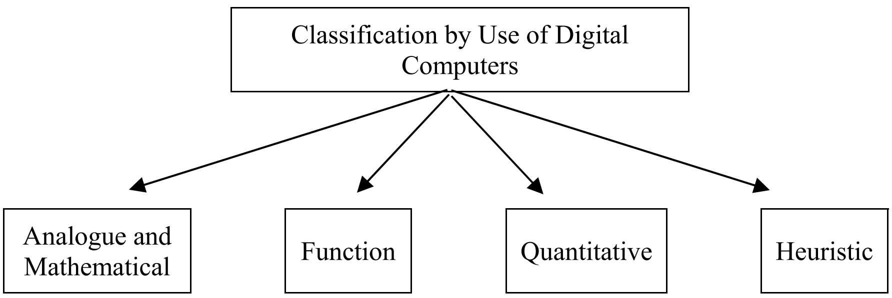
Analytical Models: These models have a specific mathematical structure and can be solved by analytical or mathematical techniques.

For example, a general linear programming, specially structured transportation and assignment models.

Simulation Models: These models also have a mathematical structure but cannot be solved by purely using the techniques and tools of mathematics.

CLASSIFICATION BY USE OF DIGITAL COMPUTERS

Figure 6



The development of the digital computer has led to the introduction of the following types of modeling in OR:

Analogue and Mathematical Model: These models are also expressed in terms of mathematical symbols.

For example, simulation model is analogue type, but mathematical formulae are also used in it.

Function Models: Such models are grouped on the basis of the function being performed.

For example, a function may serve to acquaint the scientist with such things as – tables, carrying data, a blue-print of layouts, a program representing a sequence of operations.

Quantitative Models: These models are used to measure the observations.

For example, degree of temperature, yardstick, a unit of measurement of length value, etc.

Heuristic Models: These models are mainly used to explore alternative strategies (courses of action) that were overlooked previously and do not claim to find the best solution to the problem.

GENERAL METHODS FOR SOLVING 'OPERATIONS RESEARCH' MODELS

There are three types of methods used for solving OR models. (i) Analytic method, (ii) Iterative method, and (iii) Monte-Carlo method.

ANALYTIC METHOD

If the OR model is solved by using all the tools of classical mathematics such as : differential calculus and finite differences available for this task, then such type of solutions are called analytic solutions.

Solutions of various inventory models are obtained by using analytic procedures.

ITERATIVE METHOD

The classical method fails due to the complexity of constraints or the number of variables. In such cases, iterative methods are used. This procedure starts with a trial solution and a set of rules for improving it. The trial solution is then replaced by the improved solution, and the process is repeated until either no further improvement is possible or the cost of further calculation cannot be justified.

MONTE-CARLO METHOD

Random sampling is the basis for this technique. The following steps are involved in this method:

Step 1: Draw a flow diagram of the system.

Step 2: Take correct sample observations to select suitable model for the system. Compute the probability distributions for the variables under consideration.

Step 3: Convert the probability distribution to a cumulative distribution function.

Step 4: Select a sequence of random numbers using random number tables.

Step 5: Determine the sequence of values of variables of interest with sequence of random numbers obtained in step 4.

Step 6: Construct some standard mathematical function to the values obtained in step 5.

ROLE OF COMPUTERS IN OPERATIONS RESEARCH

Computers have played a vital role in the development of OR. It would not have achieved its present position without the use of computers. The reason is that- in most of the OR techniques, computations are so complex that these techniques would be of no practical use without computers. Many large-scale applications of OR techniques, which require only a few minutes on the computer, it may take weeks, months and sometimes even years to yield the same results manually. Hence, the computer has become an essential and integral part of OR.

The following are some of the computer software packages that are useful for rapid and effective calculations:

- QSB+ (Quantitative System for Business Plus)
- QSOM(Quantitative Systems for Operations Management)
- Value STORM
- Excel
- LINDO (Linear Interactive Discrete Optimization).
- TORA.

LIMITATIONS OF OPERATIONS RESEARCH

Operations Research has certain limitations. However, these limitations are mostly related to the problems of model building and the time and money factors involved in its application rather than practical utility.

Magnitude of Computations: OR tries to find out the optimal solution taking into account all the factors. In the modern society, these factors are enormous and expressing them in quantity and establishing relationships among these require vague (tedious) calculations which can only be handled by machines.

Non-Quantifiable Factors: OR provides a solution only when all elements related to a problem can be quantified. All relevant variables do not lend themselves to quantification. Factors which cannot be quantified, find no place in OR. Models in OR do not take into account qualitative factors or emotional factors which may be quite important.

Relation between Manager and Operation Researcher: OR being a specialist’s job, requires a statistician, who might not be aware of the business problems. Similarly, a manager fails to understand the complex working of OR. Thus, there is gap between the two. The management itself may offer a lot of resistance due to conventional thinking.

Money and Time Costs: When the basic data are subjected to frequent changes, modeling in OR is a costly affair.

Implementation: Implementation of decisions is a delicate task. It must consider complexities of human relations and behavior. Some times resistance is offered only due to psychological factors.

Professional Societies

The following are some of the societies engaged in improving the methods and concepts of OR:

- The International Federation of Operational Research Society (IFORS).
- Institute for Operations Research and the Management Sciences (INFORMS).
- Operational Research Society (ORSOC).
- European Operational Research Societies (EURO).
- Australian Society for Operations Research (ASOR).
- Operational Research Society of India (ORSI).

SUMMARY

- Operation Research is defined as the science of decision-making.
- Operations Research emerged as a systematic and integrated subject at the end of World War II.
- OR is a logical and systematic approach to provide a rational basis for decision-making.
- A decision, which takes into account all the present circumstances and is considered the best one, is called optimal decision.
- The main objective of a model is to provide means for analyzing the behavior of the system for the purpose of improving its performance.
- Models can also be classified by the purpose of its utility. The purpose of a model may be descriptive, predictive or prescriptive.
- General methods used for solving OR models are – Analytic method, Iterative method, and Monte-Carlo method.
- The computer has become an essential and integral part of OR.
- Limitations of OR are mostly related to the problems of model building and the time and money factors involved in its application rather than practical utility.

Exercise

1. What is the scope of OR in industry?
2. Define Operations Research and state its assumptions.
3. Discuss the limitations of Operations Research.
4. Describe the various steps involved in OR study.
5. Describe the phases and processes of OR.
6. What are the areas of application of OR?
7. What is modeling in OR?
8. Explain the classification of models in OR environment.
9. Describe the methods of solving OR models.
10. Explain the classification of OR models by structure.
11. Explain the classification of OR models by behavior.
12. Explain the classification of OR models of purpose.

Chapter II

Formulation and Graphical Solution of Linear Programming Problem

After reading this chapter, you will be conversant with:

- Meaning of Linear Programming Problem
- Applications of Linear Programming Problem
- Advantages and Disadvantages of Linear Programming Problem
- Formulation of Linear Programming Problem
- Graphical Method

Introduction

Many business and economic situations are concerned with a problem of planning activity. In such cases, there are limited resources and the problem is to make such a use of these resources so as to yield the maximum production or to minimize the cost of production, or to give the maximum profit, etc. Such problems are referred as problems of constrained optimization. Linear Programming Problem (LPP) is a technique for determining the optimum schedule of interdependent activities in view of the available resources.

This LPP technique is designed to help managers in planning, decision-making and to allocate the resources. The management always tries to make the most effective use of organization resources. Resources include: machinery, labor, money, time, warehouse, space and raw materials. These resources may be used to produce services such as schedules for shipping, advertising policies and investment decisions etc.

All organizations have to make decisions about how to allocate their resources. There is no organization which operates permanently with unlimited resources. So, managements must continuously allocate scarce resources to achieve the organization's goals.

The word 'Programming' means 'Planning'. It refers to the process of determining a particular plan of action amongst several alternatives. The word 'Linear' indicates that all relationships involved in a particular problem are linear.

LPP is one of the most popular techniques of Operations Research. It is a mathematical technique for allotting limited resources of a firm in an optimum manner. The technique embraces almost every functional area of the business – production, finance, marketing, distribution etc. – in every type of industry. A wide variety of problems can be placed within the framework of this technique such as, to:

- decide on product quantities to maximize profit (product mix problems).
- determine the number of advertising units of different advertising media (Radio, T.V., Magazine) to ensure maximum exposures (media selection problems).
- find the quantity of components to be used in producing products at the minimum cost (alloy mix, fertilizer mix, food mix, paint mix, gasoline mix etc.)
- decide least-cost-route of transportation of units from different plants to different warehouses (transportation problems).
- establish optimal allocation of tasks to facilities (assignment problems).
- allocate order quantities, of an item, among different suppliers (assignment problems).
- select specific investments from among alternatives so as to maximize return, minimize risk (portfolio selection).
- develop a work schedule that follows a large restaurant, a hospital or a police station to meet staff needs at all hours with minimum number of employees (staffing problems).
- determine the most economic pattern and timings for flights so as to make the most efficient use of aircraft and crews (routing problems).
- find the combination of components to be produced from standardized raw material sized, (for example, paper, steel, and glass sheets in order to keep loss to minimum).
- select the shortest route for a salesman starting from a given city, visiting each of the specified cities and then returning back to the starting city (traveling salesman problems).

In this chapter, some applications and their formulation are discussed.

Terminology

The following terms are commonly used in the study of an LPP:

Decision Variable: Decision variables are the unknowns to be determined from the solution of an LPP model.

For example, decision variables in a product mix problem represent the quantities of the products to be produced, in a media selection problem they represent advertising units of different advertising media, in a diet mix problem they represent the quantities of different foods etc.

The essential requirements of the variable are:

- they should be inter-related in terms of consumption of resources.
- the relationships among the variables should be linear.

Constraint: A constraint represents a mathematical equation regarding limitations imposed by the problem characteristics. The constraints define the limits within which a solution to the problem must be found. Most often constraints represent the limits of a resource input. The constraints must be capable of being expressed in mathematical terms.

For example, assume that a company is manufacturing x and y numbers of two products '*A*' and '*B*' and each unit of these products requires respectively '*m*' hours and '*n*' hours of the machine shop capacity for which only '*t*' hours are available. Then the constraint on the production of product A and B may be expressed algebraically as:

$$mx + ny \leq t$$

Objective Function: An objective function represents the mathematical equation of the major goal of the system in terms of unknowns called decision variables. The objective function in linear programming is of optimization type – it can be maximizing profit function or minimizing cost function. The objective function, like constraints must be capable of being expressed in mathematical terms.

For example, if we assume that each unit of product A gives a profit of Rs.p_1 and each unit of product B gives profit of Rs.p_2, then the objective of the producer, say profit maximization, may be expressed in the mathematical form called objective function as under:

$$x\,p_1 + y\,p_2 = Maximum$$

The objective function is always non-negative. The coefficients associated with the variables in the objective function are constants and they represent either unit costs or unit profits of the items.

Linear Relationships: Linear programming deals with problems in which the objective function, as well as constraints, can be expressed as linear mathematical functions. Linear relationships have two properties: proportionality and divisibility.

Non-negativity Restrictions (Constraints): Essentially, the value of decision variables must be either zero or positive. Negative values of the decision variables imply negative production. Since, such a state in a real life situation is non-existent, decision variables must assume either zero or positive values. If x and y are decision variables, their non-negativity restrictions, shall be expressed as:

$$x \geq 0,\ \ y \geq 0$$

Feasible Solution: LPP helps to optimize a linear objective function subject to linear constraints of the variables. A set of values of the decision variables which satisfies all the constraints and non-negativity restriction is called feasible solution. There are usually a large number of feasible solutions to a problem.

Optimal Solution: A feasible solution which optimizes the objective function is called optimal solution. Optimal solution thus provides the best feasible choice of values which yields the highest (in case of maximization) or lowest (in case of minimization) value of the objective functions.

Stages of LPP

Each LPP involves three stages: (i) Problem Identification, (ii) Problem Formulation, and (iii) Problem Solving.

Problem Identification: Problem identification involves identification of the available alternatives, establishing the relationship between variables, specification of the constraints (i.e., available hours, space, materials, money etc.).

Problem Formulation: Problem formulation involves construction of a mathematical model from the given data. It requires identification of the decision variables, specifying the objective, setting up mathematical equations for the constraints and the objective and presenting the objective and constraints in a comprehensive form.

Problem Solving: Problem solving involves selection of the appropriate method, obtaining solution to the problem with the help of the selected method, and testing the solution for optimality.

An LPP may be solved either by the graphical method or by simplex method. Graphical method makes use of familiar graphical analysis and is used to solve a problem which involves two decision variables but any number of constraints. Simplex method is an iterative procedure where optimal solution to the problem is obtained from a series of arithmetical steps. The simplex method provides the means of solving complicated programming problems involving two or more decision variables.

Mathematical Formulation of LPP

Mathematical formulation of LPP consists of the following steps:

Step 1: Study the given problem and find the key decisions to be made

(i.e., identify unknowns called decision variables).

Step 2: Identify the variables involved and denote them by symbols

$x_j \ (j = 1, 2, ..., n)$.

Step 3: State the feasible alternatives (generally, $x_j \geq 0$, $\forall j$).

Step 4: Identify the constraints in the given problem and express them as linear inequations and/or equations.

Step 5: Identify the objective function and express it as a linear function of the decision variables.

Step 6: Express the objective function, constraints and non-negativity condition identified in steps 5, 4 and 3 in linear programming format.

General Formulation of LPP

The general formulation of the LPP can be stated as follows:

We find the values of *n* decision variables $x_1, x_2, \ldots, x_n$, to maximize or minimize the objective function

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad \ldots(1)$$

and also satisfy the *m*-constraints:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \ (\le \text{ or } = \text{ or } \ge)\, b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \ (\le \text{ or } = \text{ or } \ge)\, b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \ (\le \text{ or } = \text{ or } \ge)\, b_i \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \ (\le \text{ or } = \text{ or } \ge)\, b_m \end{array}\right\} \qquad \ldots(2)$$

where constraints may be in the form of an inequality ($\le$ or $\ge$) or even in the form of an equation (=), and finally satisfy the non-negativity restrictions

$$x_1 \ge 0, x_2 \ge 0, \ldots, x_i \ge 0, \ldots, x_n \ge 0\,. \qquad \ldots(3)$$

The values of right side parameters $b_i\,(i = 1, 2, 3, \ldots, m)$ are restricted to non-negative values only.

- The linear function $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ which is to be minimized (or maximized) is called the **objective function** of the general LPP.
- The inequations (2) are called the **constraints** of the general LPP.
- The set of inequations (3) is usually known as the set of **non-negative restrictions** of the general LPP.
- An *n*-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers which satisfies the constraints of a general LPP is called a **solution** to the general LPP.
- Any solution to a general LPP, which also satisfies the non-negative restrictions of the problem, is called a **feasible solution**.
- Any feasible solution which optimizes (minimizes or maximizes) the objective function of a general LPP is called an **optimum solution**.

Applications

Assignment Problem

Suppose we are given *m* persons, *n* jobs, and the expected productivity c_{ij} of *i*th person on the *j*th job. We want to find assignment of persons $x_{ij} \ge 0$ for all *i* and *j*, to *n* jobs so that the average productivity of person assigned is maximum, subject to the conditions:

$$\sum_{j=1}^{n} x_{ij} \le a_i \quad \text{and} \quad \sum_{i=1}^{m} x_{ij} \le b_j$$

where, a_i is the number of persons in category *i* and b_j is the number of jobs in category *j*.

Transportation Problem

We suppose that m factories (sources) supply n warehouses (destinations) with a certain product. Factor $F_i \ (i = 1, 2, ..., m)$ produces a_i units (per unit time), and warehouse $W_j \ (j = 1, 2, ..., n)$ requires b_j units. Suppose that cost of shipping from factory F_i to warehouse W_j is directly proportional to the amount shipped; and that unit cost is c_{ij}. Let the decision variables, x_{ij}, be the amount shipped from factory F_i to warehouse W_j. The objective is to determine the number of units transported from factory F_i to warehouse W_j so that the total transportation cost $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$ is minimized by satisfying supply and demand exactly.

Efficiency on Operation of System of Dams

We determine variations in water storage of dams which generate power so as to maximize the energy obtained from the entire system. The physical limitations of storage appear as inequalities.

Optimum Estimation of Executive Compensation

The objective here is to determine a consistent plan of executive compensation in an industrial concern. Salary, job ranking and the amounts of each factor required on the ranked job level are taken into consideration by constraints of LPP.

Agricultural Planning

LPP is applied for allocating the limited resources such as acreage, labor, water, supply and working capital, etc., so as to maximize the net revenue.

Military Applications

These applications involve the problem of selecting an air weapon system against gorillas so as to keep them pinned down and simultaneously minimize the amount of aviation gasoline used, a variation of transportation problem that maximizes the total tonnage of bomb dropped on a set of targets, and the problem of community defense against disaster to find the number of defense units that should be used in the attack in order to provide the required level of protection at the lowest possible cost.

Production Management

LPP can be applied in production management for determining product mix, product smoothing, and assembly time-balancing.

Marketing Management

LPP helps in analyzing the effectiveness of advertising campaign and time based on the available advertising media. It also helps traveling salesman in finding the shortest route for his tour.

Manpower Management

LPP allows the personnel manager to analyze personnel policy combinations in terms of their appropriateness for maintaining a steady-state flow of people into through and out of the firm.

Physical Distribution

LPP determines the most economic and efficient manner of locating manufacturing plants and distribution centers for physical distribution.

Besides these LPP also involves the applications in the area of administration, education, inventory control, awarding contract and capital budgeting etc.

Advantages and Disadvantages of LPP

ADVANTAGES

- Provides flexibility in analyzing a variety of multi-dimensional problems.
- Helps in making the optimum utilization of productive resources.
- Indicates how a decision maker can employ his productive factors most effectively by choosing and allocating the resources.
- Improves the quality of decision-making by replacing rules of thumb or intuition.
- Used as a very good tool for adjusting to meet changing conditions (i.e., sales, demand etc.)
- Provides practically applicable solution.
- Helps in detecting the bottlenecks in the production process.
- Provides information base from which decisions on allocation of scarce resources can be taken.
- The user of this technique becomes more objective and less subjective.

DISADVANTAGES

- Objective functions and constraints are not linear in some problems. For example, business and industrial problems.
- There is no guarantee of getting integer valued solutions. For example, number of machines, number of persons.
- Does not take into consideration the effect of time and uncertainty.
- Large-scale problems cannot be solved even though computer facility is available.
- Parameters appearing in the model are assumed to be constant. But, in real life situations they are neither constant nor deterministic.
- It deals with only single objective.

Special Cases

Every LPP does not have a unique optimal solution. Some problems may have special cases which include:

i. Redundant Constraints.
ii. Alternative Optimal Solutions.
iii. Unbounded Solution.
iv. Infeasible Solution.

Redundant Constraints

Every constraint in an LPP defines a certain portion of the boundary of the solution region. However, if any constraint does not do this (i.e., does not define the boundaries of the feasible solution), it is called a redundant constraint. A redundant constraint is not necessary for the solution to the problem and therefore can be omitted at the time of formulation of the problem.

Alternative Optimal Solution

An LPP may have multiple optimal solutions. Therefore, an alternative optimal solution is a point in the feasible region that gives the same optimal value of the objective function that is given by another feasible point.

Unbounded Solution

When an LPP solution is permitted to be infinitely large, it is known to be unbounded. In a maximization problem, unbounded solution results if all constraints are "$\geq$" type since, in such situation there will be no upper limit on the feasible region. Similarly, an unbounded solution results if all constraints are "$\leq$" type, in case of a minimization problem.

Infeasible Solution

If an LPP has no solution that satisfies all constraints, it is said to be infeasible. Infeasible solution results when the problem has not been correctly formulated.

Formulation of LPP

Many practical problems in operations research can be expressed as linear programming problems. The following are examples of such situations:

Example 1

Application of LPP on Production Planning Problem (Maximization Problem)

The Indian Electric company manufactures two popular brands of ceiling fans, Cool Home and Bahar. Each fan is processed through two main departments: machine shop, assembly and testing shop which have respectively 1200 machine hours and 1600 man-hours of available capacity per day. Each Cool Home fan requires 3 hours of capacity of machine shop and one hour capacity of assembly and testing shop. Similarly, each Bahar fan requires 2 hours for each machine shop and assembly shop capacity.

The market for the two models has been surveyed recently which suggests that a maximum of 250 Nos. of Cool Home and 200 Nos. of Bahar can be sold per day.

If the profit of fans is Rs.60 on a Cool Home and 80 on a Bahar, what quantity of each fan should be produced to maximize profit?

Formulation

The information given is summarized in the following table:

Table

Product	Hours required per unit		Market demand per day	Profit per unit (Rs.)
	Machine Shop	Assembly and Testing Shop		
Cool Home	3	1	250	60
Bahar	2	2	200	80
Available hours per day	1200	1600		

Decision Variables: The problem is to decide how many units of each fan (Cool Home and Bahar) be produced to maximize profit.

Let x = number of Cool Home fans to be produced.

y = number of Bahar fans to be produced.

Where, x and y are the decision variables.

Objective Function: The objective is to maximize total profit (Z) from the production of the two types of fans i.e., Maximize $Z = 60\,x + 80\,y$.

Constraints: The maximum amount of profit to be realized is constrained by the production capacity of machine shop, capacity of assembly and testing shop and market size.

Maximum capacity of machine shop: 1200 machine hours

Hence, $3x + 12y \le 1200$

Maximum capacity of assembly and testing shop: 1600 man hours

Hence, $1x + 2y \le 1600$

Maximum possible sale of Cool Home: 250 Nos.

Hence, $x \le 250$

Maximum possible sale of Bahar: 200 Nos.

Hence, $y \leq 200$

Hence, the LPP can be written as:

Maximize $Z = 60x + 80y$

Subject to –

$3x + 12y \leq 1200$ (Machine shop capacity constraint)

$1x + 2y \leq 1600$ (Assembly and testing shop capacity constraint)

$x \leq 250$ (Limit of demand of Cool Home)

$y \leq 200$ (Limit of demand of Bahar)

$x, y \geq 0$ (Non-negativity constraint).

Example 2

Application of LPP on Production Planning Problem (Minimization Problem)

A manufacturer intends to market a new fertilizer produced from a mixture of two ingredients *A* and *B*.

The composition of the two ingredients is as follows:

Table

Ingredients	Composition			
	Bone metal	Nitrogen	Lime	Phosphate
A	20%	30%	40%	10%
B	40%	15%	40%	5%

The management decision is that the fertilizer–

- must be sold in bags of 20 kgs
- must contain at least 25% bone metal
- must contain at least 15% nitrogen
- must contain at least 10% phosphate.

The cost of ingredients is Rs.20 per kg. for *A* and Rs.16 per kg. for *B*. Write the LPP formulation for the quantities of the ingredients to be mixed to minimize the material cost.

Formulation

Decision Variables: The problem is to find the quantity of each ingredient to put into each bag of 20 kgs to achieve the desired characteristics at minimum cost.

Let x = quantity of Ingredient A in a bag of 20 kgs.

Let y = quantity of Ingredient B in a bag of 20 kgs.

Where, x and y are decision variables.

Objective Function: The objective is to minimize cost (Z) of the ingredients, i.e., Minimize $Z = 20x + 16y$.

Constraints

i. Each bag must weigh 20 kgs.

Therefore, $x + y = 20$

ii. Bone metal content from a bag of 20 kgs, must be atleast 5 kgs. (20×0.25).

Therefore, $0.20x + 0.40y \geq 5$

By removing decimals (multiplying both sides by 100), we obtain

$20x + 40y \geq 500$

iii. Nitrogen content from bag of 20 kgs must be atleast 3 kgs. (20×0.15)

Therefore, $0.30x + 0.15y \geq 3$

By removing decimals (multiplying both sides by 100)

$30x + 15y \geq 300$

iv. Phosphate content from bag of 20 kgs must be atleast 2 kgs. (20×0.10)

Therefore, $0.10x + 0.05y \geq 2$

By removing decimals (multiplying both sides by 100)

$10x + 5y \geq 200$.

Hence, the LPP can be written as:

Minimize $Z = 20x + 16y$

Subject to –

$x + y = 20$ (Weight constraint)

$20x + 40y \geq 500$ (Bone metal constraint)

$30x + 15y \geq 300$ (Nitrogen constraint)

$10x + 5y \geq 200$ (Phosphate constraint)

$x, y \geq 0$ (Non-negativity constraint).

Example 3

Application of LPP on Media Selection Problem

An advertising company is planning a media campaign for a client, willing to spend Rs.20,00,000 to promote a new fuel economy mode of a pressure cooker. The client wishes to limit his campaign media to a daily newspaper, radio and prime time television. The agency's own research data on cost effectiveness of advertising media suggest the following:

Table

Advertising media	Cost per unit (Rs.)	Estimated number of housewives exposed to each advertising unit
Newspaper	20,000	1,00,000
Radio	40,000	5,00,000
Television	1, 00,000	10,00,000

The client wishes that atleast 50,00,000 housewives should be exposed to T.V. advertising. Also, the expense on newspaper advertising must not exceed Rs.5,00,000. Formulate the problem as an LPP.

Formulation

Decision Variable: The problem is to decide the number of advertising units of newspaper, radio and T.V. to be bought to maximize the audience exposure.

Let x = number of newspaper advertising units to be bought

y = number of radio advertising units to be bought

z = number of Prime T.V. advertising units to be bought.

Where, x, y and z are the decision variables.

Objective Function: The objective is to maximize total number of housewives (Z) to be exposed to the advertisements i.e.,

Maximize Z = 1,00,000x + 5,00,000y +10,00,000z.

Constraints

i. Maximum total expense on advertising not to exceed Rs.20, 00,000

Hence, $20,000x + 40,000y + 1,00,000z \le 20,00,000$

i.e., $x + 2y + 5z \le 100$

ii. Atleast 50, 00,000 housewives to be exposed to T.V. advertising

Hence, $10,00,000z \ge 50,00,000$

$z \ge 5$

iii. Expense on newspaper advertising not to exceed Rs.5, 00,000

Hence, $20,000x \le 5,00,000$

$x \le 25$

Hence, the LPP can be written as:

Maximize $Z = 1,00,000x + 5,00,000y + 10,00,000z$

Subject to –

$x + 2y + 5z \le 100$ (Budget constraint)

$z \ge 5$ (Female audience exposure constraint)

$x \le 25$ (Newspaper advertising expense constraint)

$x, y, z \ge 0$ (Non-negativity constraint).

Example 4

Application of LPP on Diet Problem

The goal of the diet problem is to find the cheapest combination of foods that will satisfy all the daily nutritional requirements of a person. The problem is formulated as a linear program, where the objective is to minimize cost and meet constraints which require that nutritional needs be satisfied. We include constraints that regulate the number of calories and amounts of vitamins, minerals, fats, sodium and cholesterol in the diet.

A person undergoing a diet should take N types of vitamins. One should take a quantity of b_i from each vitamin type *i*. There exist M fruits. Each fruit *j* contains $v_{j,1}, v_{j,2}, ..., v_{j,N}$ vitamins of each type. Each fruit *j* costs p_j. Assuming that it is possible to buy a fraction of a fruit, what is the cheapest combination of fruits that should be bought while keeping the diet constraints?

Formulation

Let x_j would denote the quantity bought of fruit *j*.

Minimize: $\sum_{j=1}^{M} x_j p_j$

Subject to:

$$\sum_{j=1}^{M} x_j v_{ji} \ge b_i \quad ; \forall i$$

$$x_j \ge 0, \quad \forall j$$

The solution of this linear program gives the lowest price needed to achieve the diet constraints.

Example 5

Application of LPP on Financial Planning Problem

A bank provides four kinds of loans to its personal customers and these loans yield the following annual interest rates to the bank:

- First mortgage 14%
- Second mortgage 20%
- Home improvement 20%
- Personal overdraft 10%.

The bank has a maximum foreseeable lending capability of 250 million rupees and is further constrained by the policies:

1. First mortgages must be at least 55% of all mortgages issued and at least 25% of all loans issued (in Rupees terms)
2. Second mortgages cannot exceed 25% of all loans issued (in Rupees terms)
3. To avoid public displeasure and the introduction of a new windfall tax the average interest rate on all loans must not exceed 15%.
4. Formulate the bank's loan problem as an LP so as to maximize interest income while satisfying the policy limitations.

Note here that these policy conditions, whilst potentially limiting the profit that the bank can make, also limit its exposure to risk in a particular area. It is a fundamental principle of risk reduction that risk is reduced by spreading money (appropriately) across different areas.

Solution

Decision Variable: In this problem we are interested in the amount (in Rupees) the bank has loaned to customers in each of the four different areas (not in the actual number of such loans). Hence, let

x_i = amount loaned in area i in million rupees

(Where, $i = 1$ corresponds to first mortgages, $i = 2$ to second mortgages etc.)

and note that $x_i \geq 0$ ($i = 1, 2, 3, 4$).

Note here that it is conventional in LP's to have all variables ≥ 0. Any variable (X, say) which can be positive or negative can be written as $X_1 - X_2$ (the difference of two new variables) where $X_1 \geq 0$ and $X_2 \geq 0$.

Objective Function: To maximize interest income (which is given above) i.e.,

Maximize (interest income) = $0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4$

Constraints

a. Limit on amount lent

$x_1 + x_2 + x_3 + x_4 \leq 250$

b. Policy condition 1

$x_1 \geq 0.55(x_1 + x_2)$

i.e., first mortgages ≥ 0.55(total mortgage lending) and also

$x_1 \geq 0.25(x_1 + x_2 + x_3 + x_4)$

i.e., first mortgages ≥ 0.25(total loans)

c. Policy condition 2

$x_2 \leq 0.25(x_1 + x_2 + x_3 + x_4)$

d. Policy condition 3 – we know that the total annual interest is $0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4$ on total loans of $(x_1 + x_2 + x_3 + x_4)$.

Hence the constraint relating to policy condition (3) is

$0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4 \leq 0.15(x_1 + x_2 + x_3 + x_4)$

Graphical Method

Graphical method makes use of familiar graphical analysis and is used to solve problems which involve two unknowns (i.e., two decision variables). Problems with three or more unknowns can be solved by techniques such as simplex method. Graphical method requires drawing two axes to represent two unknowns, plotting lines representing the constraints, identifying feasible region, selecting optimal vertex and evaluating coordinates of the vertex to obtain values of the unknowns.

Procedure

Simple LPP with two decision variables can be easily solved by graphical method.

Step 1: Identify decision variables.

Step 2: Set up the equation of the objective function.

Step 3: Set up equations of the constraints.

Step 4: Write the complete LPP formulation.

Step 5: Graph constraints by reducing them to equality.

Step 6: Identify feasible solution region.

Step 7: Identify farthest/nearest vertex (farthest vertex from the origin when the objective is maximization and nearest vertex from origin when objective is minimization) among vertices in the feasible solution region.

Step 8: Calculate the co-ordinates of the above vertex as the values of the decision variables (i.e. optimal solution).

Step 9: Calculate total profit/cost from the optimal solution.

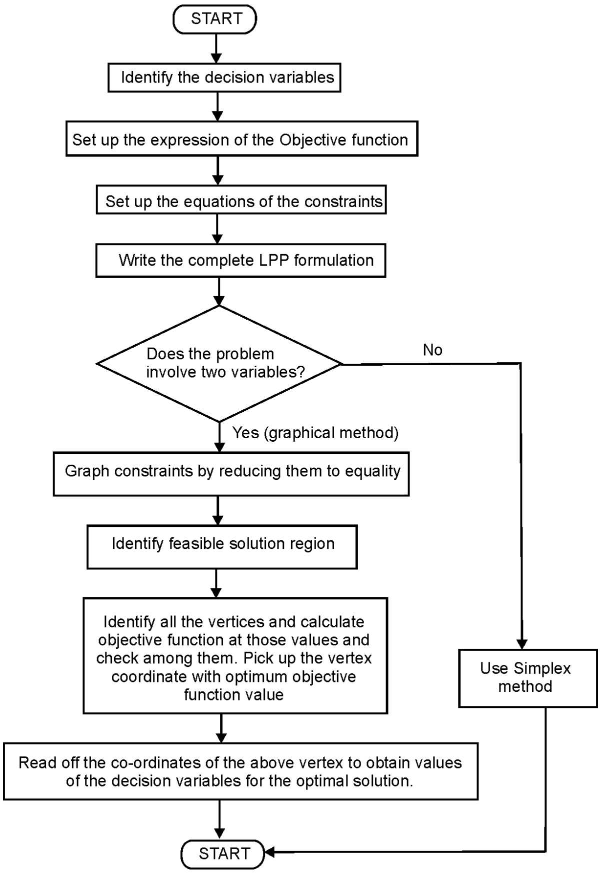
Advantages

- The graphical method is simple to understand and easy to use.
- The redundant constraints are automatically eliminated from the system.
- Multiple solutions, unbounded solutions and infeasible solutions get highlighted very clearly in the graphical analysis.
- Sensitivity analysis (a study of the effect of changes of the profit or resource level on the solution) can be easily illustrated by drawing the graph of the changes.

Disadvantages

- Graphical method is restricted to two variables. The problems involving more than two variables cannot be solved graphically.
- Though graphical method can deal with any number of constraints but since each constraint is shown as a line on graph, a large number of lines make the graph difficult to read.

Figure: Flow Chart for Graphical Method of LPP



Example 6

Application of LPP on Management Accounts

A local travel agent is planning a charter trip to a major sea resort. The eight-day, seven-night package includes the fare for round-trip travel, surface transportation, board and lodging and selected tour options. The charter trip is restricted to 200 persons and past experience indicates that there will not be any problem for getting 200 persons. The problem for the travel agent is to determine the number of Deluxe, Standard, and Economy tour packages to offer for this charter. These three plans each differ according to seating and service for the flight quality of accommodation, meal plans and tour options. The following table summarizes the estimated prices for the three packages and the corresponding expenses for the travel agent. The travel agent has hired loan air craft for the flat fee of Rs.2,00,000 for the entire trip.

Table: Table Price and Costs for Four Packages per Person

Tour Plan	Price (Rs.)	Hotel Costs (Rs.)	Meals & Other Expenses (Rs.)
Deluxe	10,000	3,000	4,750
Standard	7,000	2,200	2,500
Economy	6,500	1,900	2,200

In planning the trip, the following considerations must be taken into account:

i. At least 10 percent of the packages must be of the deluxe type.

ii. At least 35 percent but not more than 70 percent must be of the standard type.

iii. At least 30 percent must be of the economy type.

iv. The maximum number of deluxe packages available in any air craft is restricted to 60.

v. The hotel desires that at least 120 of the tourists should be on the deluxe and standard packages together.

The travel agent wishes to determine the number of packages to offer in each type so as to maximize the total profit:

a. Formulate the above as a linear programming problem.

b. Restate the above linear programming problem in terms of two decision variables, taking advantage of the fact that those 200 packages will be sold.

c. Find the optimum solution using graphical methods for the restated linear programming problem and interpret your results.

Solution

Let x_1, x_2, and x_3 be the number of Deluxe, Standard & Economy tour package restricted to 200 persons only to maximize the profits of the concern.

The contribution (per person) arising out of each type of tour package offered is as follows:

Package type offered	Price (Rs.) (1)	Hotel Costs (Rs.) (2)	Meals, etc. (Rs.) (3)	Net profit (Rs.) (4) = (1) – {(2) + (3)}
Deluxe	10,000	3,000	4,750	2,250
Standard	7,000	2,20	2,500	2,300
Economy	6,500	1,900	2,200	2,400

Since the travel agent has to pay the flat fee of Rs.2,00,000 for the chartered aircraft for the entire trip, the profit function will be:

$$\text{Max. } P = \text{Rs.}\left(2250\,x_1 + 2300x_2 + 2400x_3\right) - \text{Rs.}\,200000.$$

The constraints according to given conditions (i) to (v) are as follows:

$x_1 \geq 20$ from (i) $\quad x_3 \geq 60$ from (iii) $\quad x_1 + x_2 + x_3 = 200,$

$x_2 \geq 70$ from (ii) $\quad x_1 \leq 60$ from (iv)

$x_2 \leq 140$ from (ii) $\quad x_1 + x_2 \leq 120$ from (v)

and $x_1, x_2, x_3 \geq 0$

The above constraints can be reduced to the following compact forms:

$20 \leq x_1 \leq 60, 70 \leq x_2 \leq 140, x_3 \geq 60, x_1 + x_2 \geq 120, x_1 + x_2 + x_3 = 200, \text{and } x_1, x_2, x_3 \geq 0, \text{is}$

a. The linear programming formation is as given above.

b. Since $x_1 + x_2 + x_3 = 200$, i.e., $x_3 = 200 - (x_1 + x_2)$, substitute the value of x_3 in the above relations to get the following reduced LPP:

$\max . P = -150x_1 - 100x_2 + 280000$, subject to
$20 \leq x_1 \leq 60, 70 \leq x_2 \leq 140, 120 \leq x_1 + x_2 \leq 140 \text{ and } x_2 \geq 0.$

c. Graphical Solution

$x_1 \geq 20$ .... (1)
$x_1 \leq 60$ .... (2)
$x_2 \geq 70$ .... (3)
$x_2 \leq 140$ .... (4)
$x_1 + x_2 \leq 140$ .... (5)
$x_1 + x_2 \geq 120$ .... (6)
and x_1, $x_2 \geq 0$.

The inequalities are written in the form of equalities as:

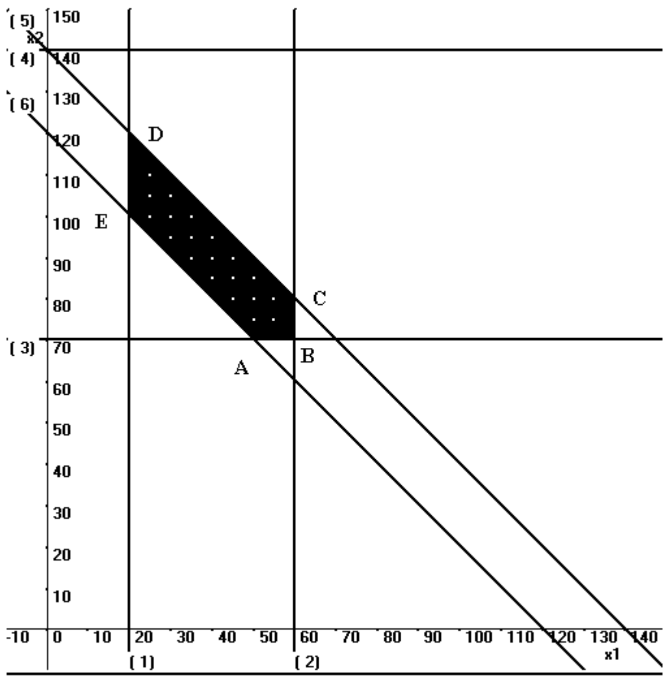
$x_1 = 20;\ x_1 = 60,$
$x_2 = 70;\ x_2 = 140,$
$x_1 + x_2 = 140;\ \ x_1 + x_2 = 120 \text{ and } x_2 = 0.$

To plot these lines we assume $x_1 = 0$ and hence obtain the value of x_2. The point is $(0, x_2)$. For the same line we assume $x_2 = 0$ and hence obtain x_1. Then, the point is $(x_1,0)$. The co-ordinates are obtained as given in the following table:

Table

Line Equation	Assumed Value (= 0)	Obtained Value	Coordinates
$x_1 + x_2 = 140$	$x_1 = 0$ $x_2 = 0$	$x_2 = 140$ $x_1 = 140$	(0,140) (140,0)
$x_1 + x_2 = 120$	$x_1 = 0$ $x_2 = 0$	$x_2 = 120$ $x_1 = 120$	(0,120) (120,0)

Figure



From the above figure, we compute,

Table

Corner points	Co-ordinates	*P*-Value
A	(50,70)	Rs.2,65,500
B	(60, 70)	Rs.2,64,000
C	(60,80)	Rs.2,63,000
D	(20, 120)	Rs.2,65,000
E	(20, 100)	Rs.2,67,000

Thus maximum profit is attained at the corner point (20, 100).

Interpretation of Solution

Maximum profit of Rs.2,67,000 is attained when $x_1 = 20$, $x_2 = 100$ and
$x_3 = 200 - (x_1 + x_2) = 80$.

In other words, the travel agent should offer 20 Deluxe, 100 Standard and 80 Economy tour packages so as to get the maximum profit of Rs.2,67,000.

Example 7

Solve the following LPP by the graphical method:

Maximize $Z = -3x_1 + 2x_2$

Subject to the constraints–

$x_1 \le 3$

$x_1 - x_2 \le 0$

and

$x_1, x_2 \ge 0.$

Solution

The two inequalities are written in the form of equalities as

$x_1 = 3$ and $x_1 - x_2 = 0$

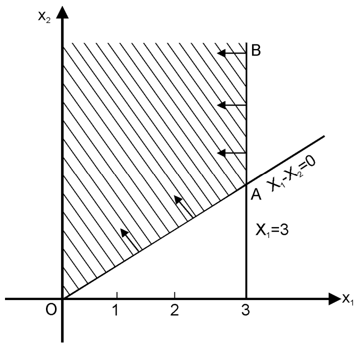
To plot these lines, we assume $x_1 = 0$ and hence obtain the value of x_2. The point is $(0, x_2)$. For the same line, we assume $x_2 = 0$ and hence obtain x_1. Then, the point is (x_1 ,0). The co-ordinates are obtained as given in the following table:

Table

Line equation	Assumed Value (= 0)	Obtained Value	Coordinates
$x_1 = 3$	Parallel to x_2-axis		$(3,0)$
$x_1 - x_2 = 0$	$x_1 = 0$	$x_2 = 0$	$(0,0)$
	$x_2 = 1$	$x_1 = 1$	$(1,1)$

Using the above co-ordinates, we graph as given below:

Figure



Conclusion

From the graph, we observe that the solution space is unbounded. It has been seen that both the variables can be made arbitrarily large as Z is increased. Here, unbounded solution does not necessarily imply that all the variables can be made arbitrarily large as Z approaches to infinity. Here, the variable x_1 remains constant.

Example 8

Solve the following LPP by graphical method:

Maximize $Z = 3x_1 - 2x_2$

Subject to constraints–

$x_1 + x_2 \le 1$

$2x_1 + 2x_2 \ge 4$

and

$x_1, x_2 \ge 0.$

Solution

The two inequalities are written in the form of equalities as:

$$x_1 + x_2 = 1 \text{ and } 2x_1 + 2x_2 = 4$$

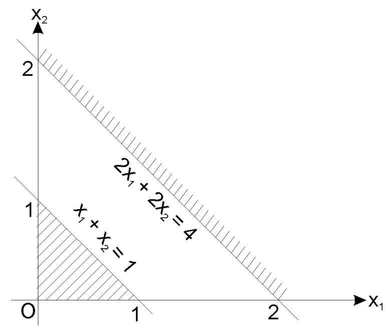
To plot these lines, we assume $x_1 = 0$ and hence obtain the value of x_2. The point is $(0, x_2)$. For the same line, we assume $x_2 = 0$ and hence obtain x_1. Then, the point is $(x_1, 0)$. The co-ordinates are obtained as given in the following table:

Table

Line Equation	Assumed Value (= 0)	Obtained Value	Coordinates
$x_1 + x_2 = 1$	$x_1 = 0$	$x_2 = 1$	(0,1)
	$x_1 = 0 = 0$	$x_1 = 1$	(1,0)
$2x_1 + 2x_2 = 4$	$x_1 = 0$	$x_2 = 2$	(0,2)
	$x_2 = 0$	$x_1 = 2$	(2,0)

Using the above co-ordinates of (x_1, x_2), we draw the lines on the figure.

Figure



Conclusion

The graph shows that there is no point (x_1, x_2) which satisfies both the constraints simultaneously. Hence, the problem has no solution because the constraints are inconsistent.

SUMMARY

- 'Linear programming' is a branch of mathematics that deals with finding extreme values of linear functions when the variables are restricted by linear inequalities.
- LPP is composed of two elements, objective function and constraints.
- 'Linear functions' are the functions in which each variable appears in a separate term is raised to the first power and is multiplied by a constant.
- 'Linear constraints' are linear functions that are restricted to be "less than or equal to", "equal to" or, "greater than or equal to" a constant.
- The maximization or minimization of some quantity is the objective in all linear programming problems.
- A 'feasible solution' satisfies all the constraints of the problem.
- An optimal solution is a feasible solution that results in the largest possible objective function value when maximizing or in the smallest possible objective function when minimizing.
- Graphical method is used to solve an LPP with two decision variables.

Exercise

1. Discuss the meaning of LPP.
2. Define Decision Variable**,** Constraint and Objective Function each with an example.
3. What are the applications of LPP?
4. Explain the advantages and disadvantages of LPP.
5. How would you formulate a LPP? Explain.
6. Describe the graphical method of solving an LPP.
7. A manufacturer of furniture makes two products, chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 5 hours on machine A and no time on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by the manufacturer from a chair and table is Rs.2 and Rs.10 respectively. What should be the daily production of each of the two products?
8. An advertising agency wishes to reach two types of audience customers with annual income greater than Rs.15,000 (target audience A) and customers with annual income less than Rs.15,000 (target audience B). The total advertising budget is Rs.2,00,000. One program of TV advertising costs Rs.50,000; one program of radio advertising costs Rs.20,000. For contract reasons, atleast 3 programs ought to be on TV and the number of radio programs must be limited to 5. Surveys indicate that a single TV program reaches 4,50,000 customers in target audience A and 50,000 in target audience B. One radio program reaches 20,000 in target audience A and 80,000 in target audience B. Determine the media mix to maximize the total reach. Solve graphically.
9. Neeti electric company produces two products P and Q on a weekly basis. The weekly production cannot exceed 25 for product P and 35 for product Q because of limited available facilities. The company employs a total of 60 workers. Product P requires 2 man weeks of labor, while Q requires one man week of labor. Profit margin on P is Rs.60 and on Q is Rs.40. Formulate it as an LPP form and solve for maximum profit.
10. A company produces two types of Hats. Each hat of the first type requires twice as much labor time as the second type. If all hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales of the first and second type to 150 and 250 hats. Assuming that the profits per hat are Rs.8 for type A and Rs.5 for type B. Formulate the problem as a linear programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.
11. A toy company manufactures two types of dolls, a basic version – doll A and deluxe version – doll B. Each doll of type B takes as long as to produce as one of type A, and the company would have time to make a maximum of 2000 per day. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which only 600 per day available. If the company makes a profit of Rs.3.00 and Rs.5.00 per doll, respectively on doll A and B, then how many of each type should be produced per day in order to maximize the total profit? Formulate this problem and solve it graphically.
12. A firm can produce three types of cloth; say A, B, C. Three kinds of wool are required for it say: red, green and blue wool. One unit length of type A cloth needs 2 meters of red wool and 3 meters of blue wool; one unit length of type B cloth needs 3 meters of red wool, 2 meters of green wool and 2 meters of blue wool; and one unit of C cloth needs 5 meters of green wool and 4 meters of blue wool. The firm has only a stock of 8 meters of red wool, 10 meters of green wool and 15 meters of blue wool. It is assumed that the income

obtained from one unit length of type A cloth is Rs.3.00 and type B cloth is Rs.5.00 and of type C cloth is Rs.4.00. Determine, how the firm should use the available material so as to maximize the income from the finished cloth.

13. A farmer has 100 acre farm. He can sell all tomatoes, lettuce, or radishes he can grow. The price he can obtain is Re.1.00 per kg for tomatoes, Rs.0.75 a head for lettuce and Rs.2.00 per kg for radishes. The average yield per-acre is 2,000 kg of tomatoes, 3000 heads of lettuce, and 1000 kgs of radishes. Fertilizer is available at Rs.0.50 per kg and the amount required per acre is 100 kgs each for tomatoes and lettuce, and 50 kgs for radishes. Labor required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes, and 6 man-days for lettuce. A total of 400 man-days of labor are available at Rs.20.00 per man-day. Formulate this problem as an LPP to maximize the farmer's total profit.

14. Solve the given LPP graphically.

    Min $Z = 5x_1 - 2x_2$

    Subject to –

    $2x_1 + 3x_2 \geq 1$

    $x_1, x_2 \geq 0.$

15. Solve the given LPP graphically.

    Max $Z = 5x_1 + 7x_2$

    Subject to –

    $x_1 + x_2 \leq 4$

    $3x_1 + 8x_2 \leq 24$

    $10x_1 + 7x_2 \leq 35$

    $x_1, x_2 \geq 0.$

Chapter III

Simplex Technique

After reading this chapter, you will be conversant with:

- Slack and Surplus Variables
- Canonical and Standard Forms LPP
- Solutions of LPP
- Simplex Algorithm
- Special Cases in Simplex Method

Introduction

Graphic method of LPP is limited to two variables only. It is an invaluable aid to understand the basic structure of an LPP. This method has limited applications in industrial problems, as the number of variables occurring there is substantially large. In such situations, we apply the method known as Simplex method, which is suitable for solving LPP for large number of variables.

Simplex method (technique) was developed by GB Dantzig, an American Mathematician, in 1947. It has the advantage of being universal i.e., any linear model for which the solution exists can be solved by this method.

Simplex method is an iterative process. Though it is an iterative process, it progressively approaches and ultimately reaches the maximum or minimum value of the objective function. This method also helps the decision maker to identify the redundant constraints, an unbounded solution, multiple solution and an infeasible solution.

For the solution of LPP by Simplex method, the objective function and the constraints are put in the form of standard mathematical model by introducing slack and surplus variables, then they are presented in a table known as simplex table and then following a procedure and rules to obtain the optimal solution improving step-by-step. Since it is an iterative procedure, each step leads closer and closer to the optimal solution. This is done by removing one basic variable at one time from the solution and replacing it by decision variables. This process is repeated till no further improvement in the solution is possible.

A general LPP is written in the following form:

Maximize/Minimize

$$z = c_1x_1 + c_2x_2 + ... + c_nx_n \qquad \ldots(1)$$

Subject to the constraints

$$\left.\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + ... + a_{1j}x_j + ... + a_{1n}x_n \left(\le \text{ or } = \text{ or } \ge\right) b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2j}x_j + ... + a_{2n}x_n \left(\le \text{ or } = \text{ or } \ge\right) b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n \left(\le \text{ or } = \text{ or } \ge\right) b_i \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mj}x_j + ... + a_{mn}x_n \left(\le \text{ or } = \text{ or } \ge\right) b_m
\end{array}\right\} \qquad \ldots(2)$$

$$x_1 \ge 0, x_2 \ge 0, ..., x_2 \ge 0, ..., x_n \ge 0 . \qquad \ldots(3)$$

SLACK AND SURPLUS VARIABLES

Slack Variable

If the constraint has $\le$ sign, then in order to make it an equality, we have to add a positive quantity to the left hand side.

The non-negative variable which is added to the left hand side of the constraint to convert it into equation is called the slack variable.

Let the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \qquad \text{for } i = 1, 2, ..., k .$$

Then, the non-negative variables x_{n+i} which satisfy

$$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i \quad i = 1, 2, ..., k$$

are called slack variables.

Example

Consider the constraints:

$x_1 + x_2 \le 2,\ 2x_1 + 4x_2 \le 5,\ \ x_1, x_2 \ge 0$

We add the slack variables $x_3 \ge 0, x_4 \ge 0$ on the left hand side of the above inequalities respectively to obtain the equations.

$x_1 + x_2 + x_3 = 2$

$2x_1 + 4x_2 + \ x_4 = 5$

$x_1, x_2, x_3, x_4 \ge 0.$

Surplus Variable

If a constraint has $\ge$ sign, then in order to make it an equality, we have to subtract a non-negative quantity from its left hand side.

Thus, the positive variable which is subtracted from the left hand side of the constraint to convert it into an equation is called the surplus variable.

Let the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij}x_j \ge b_i \qquad \text{for } i = k+1, k+2, ..., l\ .$$

Then, the non-negative variables x_{n+i} which satisfy

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i \quad i = 1, 2, ..., k$$

are called surplus variables.

Example

Consider the constraints:

$x_1 + x_2 \ge 2,\ 2x_1 + 4x_2 \ge 5,\ \ x_1, x_2 \ge 0$

We subtract the surplus variables $x_3 \ge 0, x_4 \ge 0$ on the left hand side of the above inequalities respectively to obtain the equations.

$x_1 + x_2 - x_3 = 2$

$2x_1 + 4x_2 - \ x_4 = 5$

$x_1, x_2, x_3, x_4 \ge 0.$

CANONICAL AND STANDARD FORMS OF LPP

After the formulation of LPP, we need to find the solution. To obtain the solution of any LPP, the problem must be available in a particular form. It can be (i) Canonical Form, and (ii) Standard Form.

Canonical Form

The general formulation of LPP can be expressed in the following form:

Maximize $z = c_1x_1 + c_2x_2 + ... + c_nx_n$

Subject to constraints:

$a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j ... + a_{in}x_n \le b_i \ \ i = 1, 2, ..., m$

$x_1, x_2, ..., x_n \ge 0$

by making use of some elementary transformations. This form of LPP is called the canonical form if

i. **the objective function is of the maximization type**

The minimization of a function, $f(x)$, is equivalent to the maximization

$(-f(x))$, i.e.,

Minimize $f(x) = -$Maximize $\{-f(x)\}$

For example, Max $\{-f(x)\} = -5$, Min $f(x) = -$ Max $\{-f(x)\} = -(-5) = 5$

i.e., the linear objective function

Minimize $z = c_1x_1 + c_2x_2 + ... + c_nx_n$ is equivalent to

Maximize $h = -z = -c_1x_1 - c_2x_2 - ... - c_nx_n$ with $z = -h$.

ii. **all the constraints are of the "$\le$" type, except for the non-negative restrictions**

An inequality of "$\ge$" type can be changed to an inequality of the "$\le$" type by multiplying both sides of the inequality by -1.

For example, the linear constraint

$a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n \ge b_i$ will be equivalent to

$-a_{i1}x_1 - a_{i2}x_2 - ... - a_{ij}x_j - ... - a_{in}x_n \le -b_i$

An equation may be replaced by two weak inequalities in opposite directions.

$a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n = b_i$ is equivalent to

$a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n \le b_i$ and

$a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n \ge b_i$.

iii. **all variables are non-negative**

A variable which is unrestricted in sign (i.e., positive, negative or zero) is equivalent to the difference between two non-negative variables. Thus, if x_j is unrestricted in sign, it can be replaced by $\left(x_j' - x_j''\right)$ where x_j' and x_j'' are both non-negative i.e., $x_j = x_j' - x_j''$ where $x_j' \ge 0$ and $x_j'' \ge 0$.

Standard Form

The standard form of the LPP is used to develop the procedure for solving general LPP. The characteristics of the standard form are explained in the following steps:

Step 1: *All the constraints should be written in the form of equations except for the non-negativity restrictions which remain as inequalities* (≥ 0).

Constraints of the inequality type can be changed to equations by augmenting (adding or subtracting) on the left side of each such constraint by non-negative variables.

For example,

$3x_1 - 4x_2 \ge 2,\ x_1 + 2x_2 \le 3$

These constraints can be changed to equations by introducing surplus variable x_3 and slack variable x_4 respectively.

Thus we get, $3x_1 - 4x_2 - x_3 = 2,\ x_1 + 2x_2 + x_4 = 3$ and $x_3 \ge 0, x_4 \ge 0$.

Step 2: *The right hand side element of each constraint should be made non-negative.*

The right side can always be made positive on multiplying both sides of the resulting equation by (–) sign whenever it is necessary.

For example, consider the constraints as $3x_1 - 4x_2 \geq -4$. This can be written in the form of equation by multiplying both sides by minus sign and by introducing the slack variable $x_3 \geq 0$.

Then, $-3x_1 + 4x_2 + x_3 = 4$ which is the constraint equation in standard form.

Step 3: *All variables must take non-negative values.*

A variable which is unrestricted in sign (i.e., positive, negative or zero) is equivalent to the difference between two non-negative variables. Thus, if x is unconstrained in sign, it can be replaced by $(x' - x'')$ where, x' and x'' are both non-negative i.e., $x' \geq 0$ and $x'' \geq 0$.

Step 4: *The objective function should be of maximization type.*

The minimization of a function $f(x)$ is equivalent to the maximization of the negative expression of this function, $f(x)$, i.e.,

Minimize $f(x)$ = –Maximize $\{-f(x)\}$

The linear objective function

Minimize $z = c_1x_1 + c_2x_2 + ... + c_nx_n$ is equivalent to

Max($-z$) i.e., $\text{Max}(z') = -c_1x_1 - c_2x_2 - ... - c_nx_n$ with $(-z) = z'$.

SOLUTIONS OF LPP

The following are some important definitions used for studying the solution of LPP's:

ORDINARY SOLUTION TO LPP

Any set $x = \{x_1, x_2, ..., x_{n+m}\}$ of variables is called a solution to LP problem, if it satisfies the set of constraints (2) only.

FEASIBLE SOLUTION

Any set $x = \{x_1, x_2, ..., x_{n+m}\}$ of variables is called a feasible solution (or program) of the LP problem, if it satisfies the set of constraints (2) and the non-negativity restrictions (3).

BASIC SOLUTION

Given a system of m simultaneous linear equations in n unknowns ($m < n$)

$$Ax = b, \ x^T \in R^n$$

where, A is an $m \times n$ matrix of rank m. Let B be any $m \times m$ sub-matrix formed by m linearly independent columns of A. Then a solution obtained by setting $(n - m)$ variables not associated with the columns of B, equals to zero, and solving the resulting system, is called a Basic solution to the given system of equations.

Basic Variables

The m variables which may be all different from zero, are called basic variables i.e., the variables with non-zero solution values are called basic variables. Solution with basic variables can further be divided into two categories feasible and infeasible. The $m \times m$ non-singular sub-matrix B is called basis matrix with the columns of B as basis vectors.

BASIC FEASIBLE SOLUTION (X_B)

A feasible solution to an LPP which is also a basic solution to the problem is called a basic feasible solution i.e., satisfying (3) i.e., all basic variables are non-negative.

Basic Feasible Solution is of two types:

Non-degenerate Solution: A non-degenerate basic feasible solution is the basic feasible solution which has exactly *m* positive variables, and remaining *n* variables will be zero.

Degenerate Solution: A basic feasible solution is degenerate if one or more basic variables are zero.

Note: The starting solution obtained by setting the non-basic variables to zero is Initial Basic Feasible Solution (IBFS).

OPTIMUM BASIC FEASIBLE SOLUTION

Maximize $Z = CX$

Subject to $AX = b$ and $X \geq 0$

X_B is called an optimum basic feasible solution if $Z_0 = C_B X_B \geq Z^*$ where, Z^* is the value of objective function for any feasible solution.

UNBOUNDED SOLUTION

If the value of the objective function *Z* can be increased or decreased indefinitely, such solutions are called unbounded solutions.

Non-basic Variables

The variables with zero solution values are called non-basic variables.

Example 1

Obtain all the basic solutions to the following system of linear equations:

$$x_1 + 2x_2 + x_3 = 4$$

$$2x_1 + x_2 + 5x_3 = 5.$$

Solution

Here, $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

The rank of *A* is 2. Since the rank of *A* is 2, the maximum number of linearly independent columns of *A* is 2. Thus, we can take any of the following 2×2 sub-matrices as basis matrix *B*:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}.$$

The variables not associated with the columns of B are x_3, x_2 and x_1 respectively, in the three different cases.

Case (i): Let us first take $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. A basic solution to the given system is now obtained by setting $x_3 = 0$, and solving the system

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Thus, a basic (non-basic) solution to the given system is :

Basic: $x_1 = 2,\ x_2 = 1$

Non-basic: $x_3 = 0$.

Case (ii): Taking $B = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}$. A basic solution to the given system is now obtained by setting $x_2 = 0$, and solving the system

$$\begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Thus, a basic (non-basic) solution to the given system is :

Basic: $x_1 = 5,\ x_3 = -1$

Non-basic: $x_2 = 0$.

Case (iii): Taking $B = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$. A basic solution to the given system is now obtained by setting $x_1 = 0$, and solving the system

$$\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Thus, a basic (non-basic) solution to the given system is:

Basic: $x_2 = 5/3,\ x_3 = 2/3$

Non-basic: $x_1 = 0$.

Since none of the basic variables (i.e., in all the above three basic solutions) are zero, the solutions are non-degenerate.

Example 2

Show that the following system of linear equations has a degenerate solution:

$$2x_1 + x_2 - x_3 = 2$$

$$3x_1 + 2x_2 + x_3 = 3.$$

Solution

1. The given system of equations can be written as

   $$AX = b$$

   where,

   $$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

   Since rank of A is 2, the maximum number of linearly independent columns of A is 2.

2. Thus, we can take any of the following 2×2 sub-matrices as basis matrix B:

   $$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$$

The variables not associated with the columns of B are x_3, x_1 and x_2 respectively, in the three different cases:

Case (i): Let us first take $B = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. A basic solution to the given system is now obtained by setting $x_3 = 0,$ and solving the system

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Thus, a basic (non-basic) solution to the given system is:

Basic: $x_1 = 1, x_2 = 0$

Non-basic: $x_3 = 0$.

Case (ii): Taking $B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$. A basic solution to the given system is now obtained by setting $x_1 = 0,$ and solving the system

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Thus, a basic (non-basic) solution to the given system is :

Basic: $x_2 = 5/3, x_3 = -1/3$

Non-basic: $x_1 = 0$.

Case (iii): Taking $B = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$. A basic solution to the given system is now obtained by setting $x_2 = 0,$ and solving the system

$$\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Thus, a basic (non-basic) solution to the given system is:

Basic: $x_1 = 1, x_3 = 0$

Non-basic: $x_2 = 0$.

In each of the above two basic solutions, atleast one of the basic variables is zero. Hence, solutions in case (i) and case (ii) are degenerate.

SIMPLEX ALGORITHM

Step 1: Set up the inequalities describing the problem constraints.

Step 2: Introduce slack and/or surplus variables and convert inequalities into equations.

Step 3: Obtain IBFS and construct starting simplex table.

Step 4: Calculate Z_j and value for this solution.

Step 5: Determine the entering variable by choosing the most negative $\Delta_j = Z_j - C_j$.

Step 6: Determine the row variable leaving the basis identifying key row, from the minimum value in the ratio column.

[**Note :** Compute the ratios for elements whose value in key column is > 0. Omit ratios like $\frac{25}{0} = \infty$ and $\frac{25}{-1} = -25$ etc.]

Step 7: Compute the values of the new key row using key element.

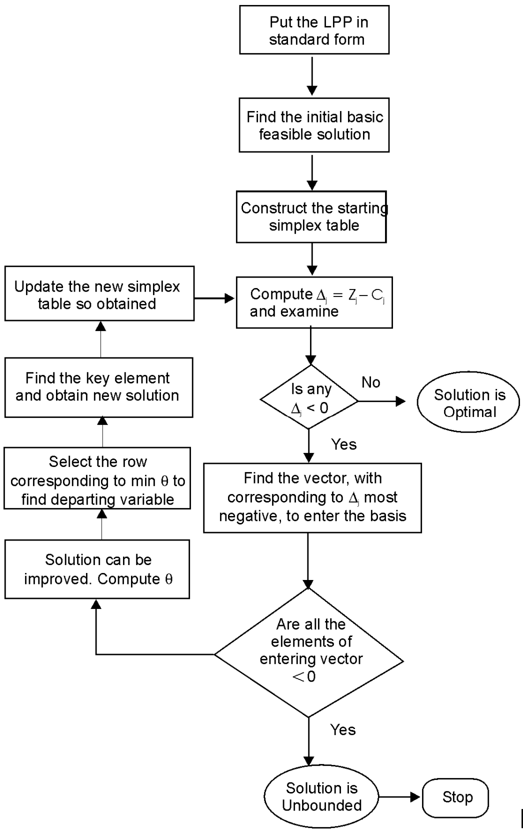
Step 8: Compute the values for the remaining rows using elementary row transformations.

Step 9: Calculate Δ_j value for this solution.

Step 10: If there is at least one negative Δ_j value, return to step 5.

Step 11: If there are no negative Δ_j values, the final solution has been obtained.

Figure: Flow Chart



Example 3

Example 3

Solve the following LPP

Maximize $Z = 6x_1 + 8x_2$

Subject to

$30x_1 + 20x_2 \leq 300$

$5x_1 + 10x_2 \leq 110$

$x_1, x_2 \geq 0.$

Solution

Convert the inequalities to equalities by adding slack variables (S_1 and S_2), since the constraints are of "$\leq$" type. The equations are:

$30x_1 + 20x_2 + S_1 = 300$

$5x_1 + 10x_2 + S_2 = 110.$

In simplex method, any unknown variable that occurs in one equation, must appear in all equations. The unknowns that do not effect an equation are written with a zero co-efficient. Here, S_1 and S_2 do not yield profit, these variables are added to the objective function with zero-coefficients. Hence, the problem is written as –

Maximize $Z = 6x_1 + 8x_2 + 0 \times S_1 + 0 \times S_2$

Subject to

$$30x_1 + 20x_2 + 1 \times S_1 + 0 \times S_2 = 300$$

$$5x_1 + 10x_2 + 0 \times S_1 + 1 \times S_2 = 110$$

and $x_1, x_2, S_1, S_2 \geq 0.$

Here, the basic variables are the slack variables S_1 and S_2 and non-basic variables are x_1 and x_2. Matrix form of constraint equations is

$$AX = B$$

where,

$$A = \begin{pmatrix} 30 & 20 & 1 & 0 \\ 5 & 10 & 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \end{pmatrix}, \quad B = \begin{pmatrix} 300 \\ 110 \end{pmatrix}.$$

IBFS

Before we go to simplex table, we need to obtain the initial solution. For this, we assume the non-basic variables equal to zero i.e., $x_1 = 0$, $x_2 = 0$. Substituting these values in the given equations, we get the values of S_1 and S_2. Hence, $S_1 = 300$ and $S_2 = 110$ are said to be in basis. Therefore, the initial solution is $S_1 = 300$ and $S_2 = 110$. Since these are ≥ 0, the solution is said to be Initial Basic Feasible Solution (IBFS).

To solve these equations we put them in the starting simplex table as follows:

Simplex Table – 1

		$C_j \rightarrow$	6	8	0	0	
C_B	Basis	X_B	x_1	x_2	S_1	S_1	θ
0	S_1	300	30	20	1	0	300/20 = 15
0	S_2	110	5	10*	0	1	110/11 = **11** $\rightarrow$
	Z_j	$Z = 0$	0	0	0	0	
		$Z_j - C_j$	–6	–8↑	0	0	

where,

C_B is the cost vector with elements as cost per unit of the variables which are in basis.

X_B is the vector with values of the variables in the basis, i.e., Initial Basic Feasible Solution (IBFS).

$$Z = \sum C_B X_B$$

C_j indicates the contribution of cost per unit of the jth variable.

$Z_j = \sum C_B X_j$ = Summation of (coefficient of C_B column × corresponding coefficients in the constraint set) i.e., Z_1 for column x_1 is $0 \times 30 + 0 \times 5 = 0$

Z_2 for column x_2 is $0 \times 20 + 0 \times 10 = 0$

Z_3 for column S_1 is $0 \times 1 + 0 \times 0 = 0$

Z_4 for column S_2 is $0 \times 0 + 0 \times 1 = 0$

$Z_j - C_j$ for column x_1 is $Z_1 - C_1 = 0 – 6 = –6$

for column x_2 is $Z_2 - C_2 = 0 - 8 = -8$

for column S_1 is $Z_3 - C_3 = 0 - 0 = 0$

for column S_2 is $Z_4 - C_4 = 0 - 0 = 0.$

By examining the $Z_j - C_j$ row, we can see that the total profit is declined by Rs.6 for each unit of x_1 added to the mix or by Rs.8 for each unit of x_2 added to the mix. Thus, a negative value in $Z_j - C_j$ row indicates that profit will be declined by the amount for each unit of x_1 or x_2. On the other hand, a positive value in the $Z_j - C_j$ row would indicate the amount by which profit would increase. Hence, the optimum solution is reached when no negative number remains in $Z_j - C_j$ row, that is more profit can be made.

Hence, $S_1 = 300$, $S_2 = 110$ is the IBFS. Now, we move towards the optimality.

***Step* 1:** Select the variable corresponding to the most negative value in $Z_j - C_j$ row. Here it is x_2. Hence, the variable x_2 should enter the basis. This column of x_2 having most negative value of $Z_j - C_j$ is known as **Key column** or **pivot column**.

***Step* 2:** Since the variable x_2 enters the basis, we must decide which variable is to be replaced (leaves the basis) as the number of variables in the basis should be same at any iteration. This can be either S_1 or S_2. For this, we apply θ*-rule* as follows:

Identify the key row by dividing each number in the solution column by the corresponding number in x_2 column (key column).

Select the row with the smaller non-negative ratio, as the row to be replaced.

Here, θ for S_1 row is $\dfrac{300}{20} = 15$

θ for S_2 row is $\dfrac{110}{11} = 10$

Since S_2 row has smaller positive value, it is called **key row**. Hence, S_2 is the departing variable.

The intersection element of key row and key column is called **key element**, indicated by * in simplex table. Here, it is in S_2 row and x_2 column with value 10.

***Step* 3:** It is decided that variable x_2 enters the basis and variable S_2 leaves basis.

Now, we have to convert the elements in x_2 column to 0's and 1 by elementary row and column transformations, key element should be made 1.

This is done by dividing every number in the key row (S_2) by the key element (i.e., 10).

$$\frac{110}{10} = 11; \ \frac{5}{10} = \frac{1}{2}; \ \frac{10}{10} = 1; \ \frac{0}{10} = 0; \ \frac{1}{10}$$

Thus, the new key row x_2 will be

$$11, \frac{1}{2}, 1, 0, \frac{1}{10}.$$

Also, replace the new values in the C_B column accordingly. Put all other values so obtained at the appropriate places.

Here, the new key row becomes

$$8, x_2, 11, \frac{1}{2}, 1, 0, \frac{1}{10}.$$

***Step* 4:** To complete the second table, we compute new values for the remaining rows. All remaining rows of the variables in the table are calculated using the formula –

$$\begin{pmatrix}\text{Element in}\\ \text{New Row}\end{pmatrix} = \begin{pmatrix}\text{Element in}\\ \text{Corresponding}\\ \text{Old Row}\end{pmatrix} - \left[(\text{FR}) \times \begin{pmatrix}\text{Corresponding}\\ \text{Element in}\\ \text{Key Row}\end{pmatrix}\right]$$

where,

$$\text{FR (Fixed Ratio)} = \frac{\text{Element in key column}}{\text{Key element}}$$

Using this formula new S_1 row is obtained as follows:

Here, FR = 20/10 = 2

Value in solution column $= 300 - 2 \times 110 = 80$

For x_1 column $= 30 - 2 \times 5 = 20$

For x_2 column $= 20 - 2 \times 10 = 0$

For S_1 column $= 1 - 2 \times 0 = 1$

For S_2 column $= 0 - 2 \times 1 = -2$

Therefore, the values for the new row become $0, S_1, 80, 20, 0, 1, -2$

***Step* 5:** The final step of the second iteration, is to compute the Z_j and $Z_j - C_j$ row for testing optimality.

Z_j are calculated as –

Z_j (total profit) $= 0 \times 80 + 8 \times 11 = 88$

i.e.,

Z_1 for column x_1 $= 0 \times 20 + 8 \times \frac{5}{10} = 4$

Z_2 for column x_2 $= 0 \times 0 + 8 \times 1 = 8$

Z_3 for column S_1 $= 0 \times 1 + 8 \times 0 = 0$

Z_4 for column S_2 $= 0 \times -2 + 8 \times \frac{1}{10} = \frac{8}{10}$

Simplex Table – 2

		$C_j \rightarrow$	6	8	0	0	
C_B	Basis	X_B	x_1	x_2	S_1	S_2	θ
0 8	S_1 x_2	80 11	**20*** 5/10	0 1	1 0	–2 1/10	80/20 = **4** → 11/(5/10) = 22
	Z_j	88	4	8	0	8/10	
		$Z_j - C_j$	**–2**↑	0	0	8/10	

From the Simplex Table – 2, it is clear that, $Z_j - C_j$ is negative for x_1 column. Hence, the variable x_1 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_1, indicating that S_1 should leave the basis.

Proceeding in the same manner, the next improved solution is given by,

Z_1 for column x_1 is $6\times1+8\times0=6$

Z_2 for column x_2 is $6\times0+8\times1=8$

Z_3 for column S_1 is $6\times1/20+8\times-1/40=1/10$

Z_4 for column S_2 is $6\times-1/10+8\times3/20=6/10$

$Z_j - C_j$ for column x_1 is $Z_1 - C_1 = 6-6=0$

for column x_2 is $Z_2 - C_2 = 8-8=0$

for column S_1 is $Z_3 - C_3 = 1/10-0=1/10$

for column S_2 is $Z_4 - C_4 = 6/10-0=6/10$.

Simplex Table – 3

		$C_j \rightarrow$	6	8	0	0
C_B	Basis	X_B	x_1	x_2	S_1	S_2
6 8	x_1 x_2	4 9	1 0	0 1	1/20 –1/40	–1/10 3/20
	Z_j	96	6	8	1/10	6/10
		$Z_j - C_j$	0	0	1/10	6/10

Conclusion

From Simplex Table – 3, we observe that all the elements in $Z_j - C_j \geq 0$, optimum solution is attained. Hence,

Optimum Solution: $x_1 = 4$ and $x_2 = 9$

Optimum Value: Maximum of $Z = 96$

Example 4

A company manufactures 3 types of parts which use precious metals, platinum and gold. Due to shortage of these precious metals, the government regulates the amount that may be used per day. The relevant data with respect to supply requirements and profit are summarized in the table shown below:

Product	Platinum per unit (gm.)	Gold required per unit (gm.)	Profit Rs. per unit (gm.)
A	2	3	500
B	4	2	600
C	6	4	1200

Daily allotments of platinum and gold are 160 gm and 120 gm. respectively. How should the company divide the supply of scarce precious metals? What is the optimum profit?

Solution

Formulation of LPP

There are three types of products *A, B* and *C*, expressed in number of units as x_1, x_2 and x_3 respectively.

Product	Platinum per unit (gm.)	Gold required per unit (gm.)	Profit per unit (gm.)
A	2	3	500
B	4	2	600
C	6	4	1200
Availability	160	120	

So,

Maximize $Z = 500x_1 + 600x_2 + 1200x_3$

Subject to –

(There are two types of constraints)

$2x_1 + 4x_2 + 6x_3 \le 160$ (platinum)

$3x_1 + 2x_2 + 4x_3 \le 120$ (gold)

$x_1, x_2, x_3 \ge 0.$

Convert the inequalities to equalities by adding slack variables S_1 and S_2, since the constraints are of "$\le$" type.

$2x_1 + 4x_2 + 6x_3 + S_1 = 160$

$3x_1 + 2x_2 + 4x_3 + S_2 = 120$

Hence, the Standard form of LPP is written as –

Maximize $Z = 500x_1 + 600x_2 + 1200x_3 + 0 \times S_1 + 0 \times S_2$

Subject to

$2x_1 + 4x_2 + 6x_3 + S_1 + 0\,S_2 = 160$

$3x_1 + 2x_2 + 4x_3 + 0\,S_1 + S_2 = 120$ and

$x_1, x_2, S_1, S_2 \ge 0$

Here, the basic variables are the slack variables S_1 and S_2 and non-basic variables are x_1 and x_2 .

Matrix form of constraint equations is

$$AX = B$$

where,

$$A = \begin{pmatrix} 2 & 4 & 6 & 1 & 0 \\ 3 & 2 & 4 & 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad B = \begin{pmatrix} 160 \\ 120 \end{pmatrix}$$

IBFS

Before we go to simplex table, we need to obtain the initial solution. For this, we assume the non-basic variables equal to zero i.e., $x_1 = 0$, $x_2 = 0$. Substituting these values in the given equations, we get the values of S_1 and S_2. Hence, $S_1 = 160$ and $S_2 = 120$ are said to be in basis. Therefore, the initial solution is $S_1 = 160$ and $S_2 = 120$. Since these are ≥ 0, the solution is said to be Initial Basic Feasible Solution.

To solve these equations we put them in the starting simplex table as follows:

Simplex Table – 4

		$C_j \rightarrow$	500	600	1200	0	0	
C_B	Basis	X_B	x_1	x_2	x_3	S_1	S_2	θ
0	S_1	160	2	4	6*	1	0	160/6 = **26.67** $\rightarrow$
0	S_2	120	3	2	4	0	1	120/4 = 30
	Z_j	0	0	0	0	0	0	
		$Z_j - C_j$	–500	–600	–1200$\uparrow$	0	0	

where,

C_B is the cost vector with elements as cost per unit of the variables which are in basis.

X_B is the vector with values of the variables in the basis, i.e., Initial Basic Feasible Solution (IBFS).

$$Z_B = \sum C_B X_B$$

C_j indicates the contribution cost per unit of the jth variable.

$$Z_j = \sum C_B X_j$$

From the Simplex Table – 4, it is clear that, $Z_j - C_j$ is minimum for x_3 column. Hence, the variable x_3 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_1, indicating that S_1 should leave the basis.

Applying elementary row and column operations and using the formula

$$\begin{pmatrix}\text{Element in}\\ \text{New Row}\end{pmatrix} = \begin{pmatrix}\text{Element in}\\ \text{Corresponding}\\ \text{Old Row}\end{pmatrix} - \left[(\text{FR}) \times \begin{pmatrix}\text{Corresponding}\\ \text{Element in}\\ \text{Key Row}\end{pmatrix}\right]$$

where,

$$\text{FR (Fixed Ratio)} = \frac{\text{Element in key column}}{\text{Key element}}$$

We have,

Simplex Table – 5

		$C_j \rightarrow$	500	600	1200	0	0	
C_B	Basis	X_B	x_1	x_2	x_3	S_1	S_2	θ
1200	x_3	160/6	1/3	2/3	1	1/6	0	80
0	S_2	80/6	**5/3***	–4/6	0	–2/3	1	**8** $\rightarrow$
	Z_j	32000	400	800	1200	200	0	
		$Z_j - C_j$	-100$\uparrow$	200	0	200	0	

From the Simplex Table – 5, it is clear that, $Z_j - C_j$ is minimum for x_1 column. Hence, the variable x_1 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_2, indicating that S_2 should leave the basis.

Simplex Table – 6

		$C_j \rightarrow$	500	600	1200	0	0
C_B	Basis	X_B	x_1	x_2	x_3	S_1	S_2
1200	x_3	24	0	4/5	1	27/90	–1/5
500	x_1	8	1	–2/5	0	–2/5	3/5
	Z_j	32800	500	760	1200	160	60
		$Z_j - C_j$	0	160	0	160	60

Conclusion

Since all $Z_j - C_j \geq 0$, optimum solution is attained. Hence,

Optimum Solution: $x_1 = 8$ and $x_3 = 24$

Optimum Value: Maximum profit $Z = 32800$

i.e., the company should supply 8 units of product A and 24 units of product B to have an optimum profit of Rs.32,800.

SPECIAL CASES IN SIMPLEX METHOD

Several complications can occur while solving LPP. Such problems are:

- Tie for the key column
- Tie for the key row (degeneracy)
- Unbounded problems

- Multiple optimal solutions
- Infeasible problems
- Redundant constraints
- Unrestricted variables.

TIE FOR THE KEY COLUMN

The non-basic variable that is selected to enter the solution is determined by the most negative value in case of maximization and largest positive value in case of minimization. This problem arises when there is tie between identical $Z_j - C_j$ values, i.e., two or more columns have exactly the same $Z_j - C_j$ value. In such a situation, selection for key column can be made arbitrarily.

TIE FOR THE KEY ROW (DEGENERACY)

Degeneracy occurs when there is tie for the minimum ratio (θ) in simplex table, for choosing the departing (leaving) variable. The main drawback of degeneracy is that it increases computation.

To resolve degeneracy, the following procedure is followed:

Step 1: Locate the rows in which smallest non-negative ratios are tied (equal).

Step 2: Find the coefficient of the slack variables and divide each coefficient by the corresponding positive numbers of the key column in the row, starting from left to right in order to break the tie by selecting minimum ratio.

Step 3: If the ratios do not break, find the similar ratios for the co-efficient of decision variables.

Step 4: Compare the resulting ratio, column by column.

Step 5: Select the row which contains smallest ratio. This row becomes the key row.

Step 6: After resolving this tie, simplex method is applied to obtain the optimum solution.

Note: If tie has occurred between artificial variable and other variable, the artificial variable should be selected as departing variable without going for the above said procedure.

Example 7

Solve the following LPP

Minimize $Z = -5x_1 - 3x_2$

Subject to –

$x_1 + x_2 \leq 2$

$5x_1 + 2x_2 \leq 10$

$-2x_1 - 8x_2 \geq -12$

$x_1,\ x_2 \geq 0$.

Solution

Multiplying both sides of the constraint, $-2x_1 - 8x_2 \geq -12$ by (–1) so that RHS is positive i.e., $2x_1 + 8x_2 \leq 12$.

Introduce slack variables S_1, S_2, S_3 with zero coefficients and convert the inequalities into equations. Hence, the standard form of LPP is –

Minimize $Z = -5x_1 - 3x_2 + 0.S_1 + 0.S_2 + 0.S_3$

Subject to –

$$x_1 + x_2 + S_1 + 0.S_2 + 0.S_3 = 2$$

$$5x_1 + 2x_2 + 0.S_1 + S_2 + 0.S_3 = 10$$

$$2x_1 + 8x_2 + 0.S_1 + 0.S_2 + S_3 = 12$$

$$x_1, x_2, S_1, S_2, S_3 \geq 0 .$$

Matrix form of constraint equations is

$$AX = B$$

where,

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 2 & 8 & 0 & 0 & 1 \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}, \; B = \begin{pmatrix} 2 \\ 10 \\ 12 \end{pmatrix}.$$

IBFS

The IBFS is obtained by considering the non-basic variables as zero i.e.,

$$x_1 = 0,\; x_2 = 0$$

Hence, the IBFS is

$$S_1 = 2,\; S_2 = 10,\; S_3 = 12$$

With the IBFS, we move towards optimality using simplex table.

Simplex Table – 7

		$C_j \rightarrow$	–5	–3	0	0	0	
C_B	Basis	X_B	x_1	x_2	S_1	S_2	S_3	θ
0	S_1	2	1	1	1	0	0	2
0	S_2	10	**5***	2	0	1	0	**2** $\rightarrow$
0	S_3	12	2	8	0	0	1	6
	Z_j	0	0	0	0	0	0	
		$Z_j - C_j$	**5** $\uparrow$	3	0	0	0	

From the Simplex Table – 7 it is clear that, $Z_j - C_j$ is maximum (since the problem is of minimization of objective function) for x_1 column. Hence, the variable x_1 should enter the basis.

Again from θ-rule, the ratio is minimum for the variables S_1 and S_2.

Here, we find there is a **tie** between two departing variables S_1 and S_2. Hence, the problem of degeneracy arises. It is resolved as follows:

	S_1	S_2	S_3	
S_1	1/1	0/1	0/1	
S_2	0/5	**1/5***	0/5	Least +ve value $\rightarrow$

Since, S_2 takes the least positive value, S_2 leaves the basis.

Simplex Table – 8

		$C_j \rightarrow$	–5	–3	0	0	0	
C_B	Basis	X_B	x_1	x_2	S_1	S_2	S_3	θ
0	S_1	0	0	**3/5***	1	–1/5	0	**0** $\rightarrow$
–5	x_1	2	1	2/5	0	1/5	0	5
0	S_3	8	0	36/5	0	–2/5	1	40/36
	Z_j	–10	–5	–2	0	–1	0	
		$Z_j - C_j$	0	**1** $\uparrow$	0	1	0	

From the Simplex Table – 8, it is clear that, $Z_j - C_j$ is maximum for x_2 column. Hence, the variable x_2 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_1, indicating that S_1 should leave the basis.

Simplex Table – 9

		$C_j \rightarrow$	–5	–3	0	0	0
C_B	Basis	X_B	x_1	x_2	S_1	S_2	S_3
–3	x_2	0	0	1	5/3	–1/3	0
–5	x_1	2	1	0	–2/3	1/3	0
0	S_3	8	0	0	12	2	1
	Z_j	–10	–5	–3	–5/3	–2/3	0
		$Z_j - C_j$	0	0	–5/3	–2/3	0

Conclusion

From Simplex Table – 9, it is observed that all $Z_j - C_j$ values are ≤ 0. Hence, the current solution is optimal.

Optimal Solution: $x_1 = 2$, $x_2 = 0$,

Optimal Value: Minimum of $Z = -10$.

UNBOUNDED PROBLEM

The case of unbounded solution occurs when the feasible region is unbounded such that the value of the objective function can be increased indefinitely. It is not necessary that an unbounded feasible region should yield an unbounded value for the objective function.

Example 8

Solve the following LPP

Maximize $Z = 107x_1 + x_2 + 2x_3$

Subject to –

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \leq 5$$

$$3x_1 - x_2 - x_3 \leq 0$$

$$x_1,\ x_2, x_3, x_4 \geq 0.$$

Solution

By introducing slack variables S_1, S_2, the set of constraints is converted into system of equations as:

$$\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 = \frac{7}{3}$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 + S_1 = 5$$

$$3x_1 - x_2 - x_3 + S_2 = 0$$

Note: Here the original variable x_4 has been treated as slack variable as its coefficient in the objective function is zero.

Then, the standard form of LPP is given by–

Maximize $Z = 107x_1 + x_2 + 2x_3 + 0S_1 + 0S_2$

Subject to–

$$\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 + 0S_1 + 0S_2 = \frac{7}{3}$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 + 0x_4 + S_1 + 0S_2 = 5$$

$$3x_1 - x_2 - x_3 + 0x_4 + 0S_1 + S_2 = 0$$

$$x_1, x_2, x_3, x_4, S_1, S_2 \geq 0$$

Matrix form of constraint equations is

$$AX = B$$

where,

$$A = \begin{pmatrix} 14/3 & 1/3 & -2 & 1 & 0 & 0 \\ 16 & 1/2 & -6 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{pmatrix}, \ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ S_1 \\ S_2 \end{pmatrix}, \quad B = \begin{pmatrix} 7/3 \\ 5 \\ 0 \end{pmatrix}.$$

IBFS

The IBFS is obtained by considering the non-basic variables as zero i.e.,

$$x_1 = 0,\ x_2 = 0, x_3 = 0.$$

Hence, the IBFS is

$$x_4 = 7,\ S_1 = 5,\ S_2 = 0.$$

With the IBFS, we move towards optimality using simplex table.

Simplex Table – 10

		$C_j \rightarrow$	107	1	2	0	0	0	
C_B	Basis	X_B	x_1	x_2	x_3	x_4	S_1	S_2	θ
0	x_4	7/3	14/3	1/3	–2	1	0	0	7/14
0	S_1	5	16	1/2	–6	0	1	0	5/16
0	S_2	0	**3***	–1	–1	0	0	1	**0/3** $\rightarrow$
	Z_j	0	0	0	0	0	0	0	
		$Z_j - C_j$	**–107** $\uparrow$	–1	–2	0	0	0	

From the Simplex Table – 10, it is clear that, $Z_j - C_j$ is minimum for x_1 column. Hence, the variable x_1 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_2, indicating that S_2 should leave the basis.

Simplex Table – 11

		$C_j \rightarrow$	107	1	2	0	0	0	
C_B	Basis	X_B	x_1	x_2	x_3	x_4	S_1	S_2	θ
0	x_4	7/3	0	17/9	– **4/9**	1	0	–14/9	–
0	S_1	5	0	35/6	– **2/3**	0	1	–16/3	–
107	x_1	0	1	–1/3	– **1/3**	0	0	1/3	–
	Z_j	0	107	–107/3	–107/3	0	0	107/3	
		$Z_j - C_j$	0	–110/3	–113/3 ↑	0	0	–107/3	

Conclusion

From the Simplex Table – 11, it is clear that, $Z_j - C_j$ is minimum for x_3 column. Hence, the variable x_3 should enter the basis.

Again from θ-rule, the ratios cannot be determined since, all the elements of key column are negative.

Therefore, we cannot decide which variable should leave the basis though the entering variable is x_3. Such a problem is said to have an unbounded solution.

MULTIPLE OPTIMAL SOLUTION

In the final simplex table, if the key row indicates the value of $Z_j - C_j$ for a non-basic variable to be zero, there exists an alternative optimum solution. This is irrespective of whether the variable is a decision or slack or surplus variable. To find the alternative optimal solution (s), the non-basic variable with the $Z_j - C_j$ value of zero should be selected as the entering variable and the simplex steps are continued.

Suppose X_1 is one optimum solution and X_2 is another optimum solution, then the multiple optimum solutions X can be expressed as a linear combination as follows:

$$X = \lambda X_1 + (1-\lambda) X_2 \quad \text{for } 0 \leq \lambda \leq 1.$$

Example 9

Solve the following LPP

Maximize $Z = 2000x_1 + 3000x_2$

Subject to –

$$6x_1 + 9x_2 \leq 100$$

$$2x_1 + x_2 \leq 20$$

$$x_1, x_2 \geq 0.$$

Solution

By introducing slack variables S_1, S_2, the set of constraints is converted into a system of equations as –

$$6x_1 + 9x_2 + S_1 = 100$$

$$2x_1 + x_2 + S_2 = 20$$

Hence the standard form of LPP is given as follows–

Maximize $Z = 2000x_1 + 3000x_2 + 0S_1 + 0S_2$

Subject to–

$6x_1 + 9x_2 + S_1 + 0S_2 = 100$

$2x_1 + x_2 + 0S_1 + S_2 = 20$

$x_1, x_2, S_1, S_2 \geq 0$.

Matrix form of constraint equations is

$AX = B$

where,

$$A = \begin{pmatrix} 6 & 9 & 1 & 0 \\ 9 & 1 & 0 & 1 \end{pmatrix}, X \begin{pmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \end{pmatrix} \quad B = \begin{pmatrix} 100 \\ 20 \end{pmatrix}.$$

IBFS

The IBFS is obtained by considering the non-basic variables as zero i.e.,

$x_1 = 0,\ x_2 = 0$

Hence, the IBFS is

$S_1 = 100,\ S_2 = 20$

With the IBFS, we move towards optimality using simplex table.

Simplex Table – 12

		$C_j \to$	2000	3000	0	0	
C_B	Basis	X_B	x_1	x_2	S_1	S_2	θ
0	S_1	100	6	**9***	1	0	**100/9** $\to$
0	S_2	20	2	1	0	1	20/1
	Z_j	0	0	0	0	0	
		$Z_j - C_j$	–2000	**–3000** $\uparrow$	0	0	

From the Simplex Table – 12, it is clear that, $Z_j - C_j$ is minimum for x_2 column. Hence, the variable x_2 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_1, indicating that S_1 should leave the basis.

Simplex Table – 13

		$C_j \to$	2000	3000	0	0
C_B	Basis	X_B	x_1	x_2	S_1	S_2
3000	x_2	100/9	2/3	1	1/9	0
0	S_2	80/9	4/3	0	–1/9	1
	Z_j	100000/3	2000	3000	1000/3	0
		$Z_j - C_j$	0	0	1000/3	0

It is observed from the Simplex Table – 13 that all $Z_j - C_j$ values are ≥ 0. Hence, the current solution is optimal.

Optimal Solution: $x_1 = 0$, $x_2 = 100/9$.

Optimal Value: Maximize $Z = 100000/3$.

To find Alternate Optima

Recalling that $Z_j - C_j$ value indicates the per unit net increase in profit would be realized from entering a non-basic variable; we can see that entering variable x_1 would neither decrease nor increase profit. It would result in a different solution having same Z_j. In order to compute the value of the alternative optimum solution we introduce x_1 as a basic variable, replacing S_2. The resultant simplex table is given as follows:

Simplex Table – 14

		$C_j \rightarrow$	2000	3000	0	0
C_B	Basis	X_B	x_1	x_2	S_1	S_2
3000	x_2	20/3	0	1	1/6	1/2
2000	x_1	20/3	1	0	–1/12	3/4
	Z_j	100000/3	2000	3000	1000/3	0
		$Z_j - C_j$	0	0	1000/3	0

Conclusion

It is observed from Simplex Table – 14, that all $Z_j - C_j$ values are ≥ 0. Hence, the current solution is optimal.

Optimal Solution:

$$X_1 = \begin{pmatrix} 0 \\ 100/9 \end{pmatrix} \quad \text{or} \quad X_2 = \begin{pmatrix} 20/3 \\ 20/3 \end{pmatrix}$$

And the set of multiple optima can be expressed as:

$$X = \lambda X_1 + (1-\lambda) X_2 \quad \text{for } 0 \leq \lambda \leq 1.$$

$$\text{i.e., } X = \lambda \begin{pmatrix} 0 \\ 100/9 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 20/3 \\ 20/3 \end{pmatrix} = \begin{pmatrix} (1-\lambda) 20/3 \\ 1/9(60+40\lambda) \end{pmatrix}.$$

Optimal Value: $Z = 100000/3$.

PROBLEM WITH INFEASIBLE SOLUTION

This condition occurs when the problem has incompatible constraints. The final simplex table shows optimal solution when all $Z_j - C_j$ elements are ≥ 0 in case of maximization and ≤ 0 in case of minimization. However, observing the solution base, we find that an artificial variable is present in basis. Both these values are totally meaningless since the artificial variable has no meaning. Hence, in such a situation, LPP is said to have an infeasible solution.

REDUNDANT CONSTRAINTS

Consider the constraints

$$3x_1 + 4x_2 \leq 7$$

$$3x_1 + 4x_2 \leq 15.$$

It is clear that second constraint is less restrictive (because both the constraints have same coefficients and variable) than the first one, and is not required. Normally, redundant constraint does not pose any problem except that the computational work is unnecessarily increased.

UNRESTRICTED VARIABLE

Unrestricted variable is that decision variable which can carry any value. That is, this variable can take positive or negative or zero values. To solve the problem by Simplex, we express it in the standard form. In LPP there is non-negative restriction $x_j \geq 0$. To satisfy this condition an unrestricted variable (x_j) can be expressed as the difference of two non-negative values $(x'_j - x''_j)$ where, $x'_j, x''_j \geq 0$. Now all the variables become non-negative in the system and the problem is solved. After obtaining the answer, the value of original unrestricted variable is obtained by taking it as the difference between two non-negative values.

SUMMARY

- Simplex method is suitable for solving LPP containing large number of variables.
- Simplex method is an iterative process.
- For the solution of LPP by Simplex method, the objective function and the constraints are put in the form of standard mathematical model, then they are presented in a table known as simplex table and then following a procedure and rules to obtain the optimal solution improving step by step. Since it is an iterative procedure, each step leads closer and closer to the optimal solution.
- Simplex method also helps the decision maker to identify the redundant constraints, an unbounded solution, multiple solution and an infeasible solution.
- Degeneracy occurs when there is tie for the minimum ratio (θ) in simplex table, for choosing the departing (leaving) variable.
- The case of unbounded solution occurs when the feasible region is unbounded such that the value of the objective function can be increased indefinitely.
- In the final simplex table, if the key row indicates the value of $Z_j - C_j$ for a non-basic variable to be zero, there exists an alternative optimum solution.

Exercise

1. Explain various steps of the simplex method involved in the computation of an optimum solution to a linear programming problem.
2. Give outlines of the Simplex in Linear Programming.
3. What is Simplex? Describe the simplex method of solving LPP.
4. What is the significance of $Z_j - C_j$ numbers in the simplex table? Interpret their economic significance in terms of marginal worth.
5. How do graphical and simplex methods of solving LPP differ? In what ways they are same?
6. How do maximization and minimization problems differ when applying the simplex method?
7. What is the reason behind the use of minimum ratio test in selecting the key row? What might happen without it?
8. What conditions exists in a simplex table to establish the existence of an alternative solution? No feasible solution? Unbounded solution ? Degeneracy?

   Solve the following problems using Simplex method.

9. Maximize $Z = 5x_1 + 3x_2$

   Subject to

   $3x_1 + 5x_2 \leq 15$

   $5x_1 + 2x_2 \leq 10$

   $x_1, x_2 \geq 0.$

10. Maximize $Z = 7x_1 + 5x_2$

    Subject to

    $-x_1 - 2x_2 \geq -6$

    $4x_1 + 3x_2 \leq 12$

    $x_1, x_2 \geq 0.$

11. Maximize $Z = 2x_1 + 4x_2$

    Subject to

    $2x_1 + 3x_2 \leq 48$

    $x_1 + 3x_2 \leq 42$

    $x_1 + x_2 \leq 21$

    $x_1, x_2 \geq 0.$

12. Minimize $Z = -3x_1 - 4x_2$

    Subject to

    $x_1 - x_2 \leq 1$

    $-x_1 + x_2 \leq 2$

    $x_1, x_2 \geq 0.$

13. Minimize $Z = -x_1 - x_2 - x_3$

    Subject to

    $4x_1 + 5x_2 + 3x_3 \le 15$

    $10x_1 + 7x_2 + x_3 \le 12$

    $x_1, x_2, x_3 \ge 0.$

14. Minimize $Z = -4x_1 - 5x_2 - 9x_3 - 11x_4$

    Subject to

    $x_1 + x_2 + x_3 + x_4 \le 15$

    $7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120$

    $3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100$

    $x_1, x_2, x_3, x_4 \ge 0.$

15. Given a general LPP, explain how you would test whether a basic feasible solution is an optimal solution or not. How would you proceed to change the basic feasible solution in case it is not optimal?
16. Explain the meaning of basic feasible solution and degenerate solution in a LPP.
17. What do you mean by optimum basic feasible solution in LPP?
18. Explain, what is meant by degeneracy and cycling in LPP. How can these problems be resolved?
19. Explain the term artificial variable and its use in Linear Programming.
20. Define Slack and Surplus variables in LPP.

Chapter IV

Artificial Variable Technique

After reading this chapter, you will be conversant with:

- Big-M Method
- Two-Phase Method
- Advantages of Two-Phase Method over Big-M Method
- Two-Phase Simplex Algorithm
- Flow Chart

Introduction

Simplex method is applicable only under essentially the coefficient matrix obtained from conditions of linear programming problem, must contain a basis matrix which is the identity matrix of order $m \times m$. But this identity matrix may not be directly possible to contain by a coefficient matrix when at least one constraint has >= or = sign in a LPP. In such cases artificial variables are introduced to form an identity matrix. An artificial variable is a non-negative quantity added to the left side of a constraint of an LP problem in standard form. Sufficient artificial variables are added to the constraints so that the resulting coefficient matrix contains an $m \times m$ identity sub-matrix. Although they are basic variables initially, when we apply the Simplex method, artificial variables must be driven out of the basis so that their values are zero in the final solution. If this is not possible, then the problem is infeasible (i.e., its feasible region has no extreme points). There are two standard methods for handling artificial variables within the Simplex method:

- Big-M Method
- Two-Phase Method.

Although they seem to be different, they are essentially identical. However, methodologically the Two-Phase method is much superior.

BIG-M METHOD

The Big-M method is a method of solving a linear programming problem involving artificial variables. In this method, we assign a very high penalty (say M) to the artificial variable in the objective function.

ALGORITHM

The following steps are involved in the Big-M method:

Step 1: Write the given LPP into its standard form and check whether there exists a starting basic feasible solution.

If there is a ready starting basic feasible solution, move on to Step 3.

If there does not exist a ready starting basic feasible solution, move on to Step 2.

Step 2: Add artificial variables to the left side of each equation that has no starting basic variable. Assign a very high penalty (say M) to these variables in the objective function.

Step 3: Apply simplex method to the modified LPP. Following cases may arise in the final iteration:

i. At least one artificial variable is present in the basis with zero value. In such a case, the current optimum basic feasible solution is degenerate.

ii. At least one artificial variable is present in the basis with a positive value. In such a case, the given LPP does not possess an optimum basic feasible solution (since the penalty M of the artificial variable in the objective function effects the solution). The given problem is said to have pseudo-optimum basic feasible solution.

Example 1

Solve the following LPP

Maximize $Z = x_1 + 2x_2 + 3x_3$

Subject to –

$x_1 - x_2 + x_3 \geq 4$

$x_1 + x_2 + 2x_3 \leq 8$

$x_1 - x_3 \geq 2$

$x_1, x_2, x_3 \geq 0.$

Solution

This is the problem of mixed constraints. Here we introduce slack, surplus and artificial variables.

For the inequalities of "$\le$" type we add slack variables and for the inequalities of "$\ge$" type we subtract surplus and add artificial variables. Hence, the constraints become

$$x_1 - x_2 + x_3 - S_1 + A_1 = 4$$

$$x_1 + x_2 + 2x_3 + S_2 = 8$$

$$x_1 - x_3 - S_3 + A_2 = 2$$

The standard form of LPP is –

Maximize $Z = x_1 + 2x_2 + 3x_3 + 0.S_1 + 0.S_2 + 0.S_3 - MA_1 - MA_2$

Subject to –

$$x_1 - x_2 + x_3 - S_1 + 0.S_2 + 0.S_3 + A_1 + 0.A_2 = 4$$

$$x_1 + x_2 + 2x_3 + 0.S_1 + S_2 + 0.S_3 + 0.A_1 + 0.A_2 = 8$$

$$x_1 + 0.x_2 - x_3 + 0.S_1 + 0.S_2 - S_3 + 0.A_1 + A_2 = 2$$

$$x_1, x_2, x_3, S_1, S_2, S_3, A_1, A_2 \ge 0.$$

Matrix form of constraint equations is

$$AX = B$$

where,

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ S_1 \\ S_2 \\ S_3 \\ A_1 \\ A_2 \end{pmatrix}, \; B = \begin{pmatrix} 4 \\ 8 \\ 2 \end{pmatrix}$$

IBFS

The IBFS is obtained by setting the non-basic variables to zero i.e.,

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ S_1 = 0,\ S_3 = 0$$

Hence, the IBFS is

$$A_1 = 4,\ S_2 = 8,\ A_2 = 2$$

With the IBFS, we move towards optimality using simplex table.

Simplex Table – 1

		$C_j \rightarrow$	1	2	3	0	0	0	–M	–M	
C_B	Basis	X_B	x_1	x_2	x_3	S_1	S_2	S_3	A_1	A_2	θ
–M	A_1	4	1	–1	1	–1	0	0	1	0	4/1
0	S_2	8	1	1	2	0	1	0	0	0	8/1
–M	A_2	2	**1***	0	–1	0	0	–1	0	1	**2/1** $\rightarrow$
	Z_j	–6M	–2M	M	0	M	0	M	–M	–M	
		$Z_j - C_j$	**– 1 – 2M** $\uparrow$	– 2 + M	– 3	M	0	M	0	0	

where,

C_B is the cost vector with elements as cost per unit of the variables which are in basis.

X_B is the vector with values of the variables in the basis i.e., Initial Basic Feasible Solution (IBFS).

C_j indicates the contribution cost per unit of the jth variable.

$$Z_j = \sum C_B X_j = \text{Summation of (coefficient of } C_B \text{ column}$$

$$\times \text{ corresponding coefficients in the constraint set)}$$

From the Simplex Table – 1, it is clear that, $Z_j - C_j$ is minimum* for x_1 column. Hence, the variable x_1 should enter the basis.

***Remark:** The minimum or maximum value of $Z_j - C_j$ is selected on the basis of coefficient of M in each term. If a tie occurs among coefficients, then the minimum or maximum value of $Z_j - C_j$ is selected on taking some positive integer value for M.

Again from θ-rule, the ratio is minimum for the variable A_2, indicating that A_2 should leave the basis.

Applying elementary row and column operations and using the formula

$$\begin{pmatrix}\text{Element in}\\ \text{New Row}\end{pmatrix} = \begin{pmatrix}\text{Element in}\\ \text{Corresponding}\\ \text{Old Row}\end{pmatrix} - \left[(\text{FR}) \times \begin{pmatrix}\text{Corresponding}\\ \text{Element in}\\ \text{Key Row}\end{pmatrix}\right]$$

where,

$$\text{FR (Fixed Ratio)} = \frac{\text{Element in key column}}{\text{Key element}}$$

Then, we have

Simplex Table – 2

		$C_j \rightarrow$	1	2	3	0	0	0	–M	
C_B	Basis	X_B	x_1	x_2	x_3	S_1	S_2	S_3	A_1	θ
–M	A_1	2	0	–1	**2***	–1	0	1	1	**1** $\rightarrow$
0	S_2	6	0	1	3	0	1	1	0	2
1	x_1	2	1	0	–1	0	0	1	0	–
	Z_j	2-2M	1	M	–1–2M	M	0	–M–1	–M	
		$Z_j - C_j$	0	–2+M	**–2M –4**$\uparrow$	M	0	–M –1	0	

From the Simplex Table – 2, it is clear that, $Z_j - C_j$ is minimum for x_3 column. Hence, the variable x_3 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable A_1, indicating that A_1 should leave the basis.

Simplex Table – 3

		$C_j \rightarrow$	1	2	3	0	0	0	
C_B	Basis	X_B	x_1	x_2	x_3	S_1	S_2	S_3	θ
3	x_3	1	0	–1/2	1	–1/2	0	1/2	–
0	S_2	3	0	**5/2***	0	3/2	1	–1/2	**6/5→**
1	x_1	3	1	–1/2	0	–1/2	–1/2	0	–
	Z_j	6	1	–2	3	–2	0	1	
		$Z_j - C_j$	0	**–4** ↑	0	–2	0	1	

From the Simplex Table – 3, it is clear that, $Z_j - C_j$ is minimum for x_2 column. Hence, the variable x_2 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_2, indicating that S_2 should leave the basis.

Simplex Table – 4

		$C_j \rightarrow$	1	2	3	0	0	0
C_B	Basis	X_B	x_1	x_2	x_3	S_1	S_2	S_3
3	x_3	8/5	0	0	1	–1/5	1/5	2/5
2	x_2	6/5	0	1	0	3/5	2/5	–1/5
1	x_1	18/5	1	0	0	–1/5	1/5	–3/5
	Z_j	54/5	1	2	3	2/5	8/5	1/5
		$Z_j - C_j$	0	0	0	2/5	8/5	1/5

Conclusion

From the Simplex Table – 4 we observe that all $Z_j - C_j$ values are ≥ 0. Hence, the current solution is optimal.

Optimal Solution: x_1 =18/5, x_2 = 6/5, x_3 = 8/5

Optimal Value: Maximum value of Z = 54/5.

Example 2

Solve the following LPP

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to –

$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

Solution

This problem is given in the standard form. To get IBFS, we introduce artificial variables A_1, A_2 in the first two constraints, in order to get identity matrix and assign (–M) coefficient in the objective function. In the third constraint there is variable x_4 with coefficient 1 and with zero coefficients in other constraints. Hence, the LPP is given as –

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

Subject to –

$$x_1 + 2x_2 + 3x_3 + 0.x_4 + A_1 + 0.A_2 = 15$$

$$2x_1 + x_2 + 5x_3 + 0.x_4 + 0.A_1 + A_2 = 20$$

$$x_1 + 2x_2 + x_3 + x_4 + 0.A_1 + 0.A_2 = 10$$

$$x_1, x_2, x_3, x_4, A_1, A_2 \geq 0.$$

[**Note :** The third equation does not require an artificial variable since the vector x4 has the unit vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$]

Matrix form of constraint equations is

$$AX = B$$

where,

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 1 & 5 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 & 0 \end{pmatrix}, \ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ A_1 \\ A_2 \end{pmatrix}, \ B = \begin{pmatrix} 15 \\ 20 \\ 10 \end{pmatrix}.$$

IBFS

The IBFS is obtained by setting the non-basic variables to zero i.e.,

$$x_1 = 0, \ x_2 = 0, \ \ x_3 = 0$$

Hence, the IBFS is

$$A_1 = 15, \ S_2 = 20, \ x_4 = 10$$

With the IBFS, we move towards optimality using simplex table.

Simplex Table – 5

		$C_j \rightarrow$	1	2	3	–1	–M	–M	
C_B	Basis	X_B	x_1	x_2	x_3	x_4	A_1	A_2	θ
–M	A_1	15	1	2	3	0	1	0	5
–M	A_2	20	2	1	**5***	0	0	1	**4 →**
–1	x_4	10	1	2	1	1	0	0	10
	Z_j	–35M–10	–3M–1	–3M–2	–8M–1	–1	–M	–M	
		$Z_j - C_j$	–3M –2	–3M –4	**–8M –4 ↑**	0	0	0	

From the Simplex Table – 5, it is clear that, $Z_j - C_j$ is minimum for x_3 column. Hence, the variable x_3 should enter the basis.

Again from θ-*rule* rule, the ratio is minimum for the variable A_2, indicating that A_2 should leave the basis.

Simplex Table – 6

		$C_j \rightarrow$	1	2	3	–1	–M	
C_B	Basis	X_B	x_1	x_2	x_3	x_4	A_1	θ
–M	A_1	3	–1/5	**7/5***	0	0	1	**15/7** $\rightarrow$
3	x_3	4	2/5	1/5	1	0	0	20
–1	x_4	6	3/5	9/5	0	1	0	10/3
	Z_j	6–3M	$\frac{M+3}{5}$	$\frac{-7M-6}{5}$	3	–1	–M	
		$Z_j - C_j$	$\frac{M-2}{5}$	$\mathbf{\frac{-7M-16}{5}} \uparrow$	0	0	0	

From the Simplex Table – 6, it is clear that, $Z_j - C_j$ is minimum for x_2 column. Hence, the variable x_2 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable A_1, indicating that A_1 should leave the basis.

Simplex Table – 7

		$C_j \rightarrow$	1	2	3	–1	
C_B	Basis	X_B	x_1	x_2	x_3	x_4	θ
2	x_2	15/7	–1/7	1	0	0	–
3	x_3	25/7	3/7	0	1	0	25/3
–1	x_4	15/7	**6/7***	0	0	1	**5/2** $\rightarrow$
	Z_j	90/7	1/7	2	3	–1	
		$Z_j - C_j$	**–6/7** $\uparrow$	0	0	0	

From the Simplex Table – 7, it is clear that, $Z_j - C_j$ is minimum for x_1 column. Hence, the variable x_1 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable x_4, indicating that x_4 should leave the basis.

Simplex Table – 8

		$C_j \rightarrow$	1	2	3	–1
C_B	Basis	X_B	x_1	x_2	x_3	x_4
2	x_2	5/2	0	1	0	1/6
3	x_3	5/2	0	0	1	–1/2
1	x_1	5/2	1	0	0	7/6
	Z_j	15	1	2	3	0
		$Z_j - C_j$	0	0	0	1

Conclusion

From the Simplex Table 8 it is observed that all $Z_j - C_j$ values are ≥ 0. Hence, the current solution is optimal.

Optimal Solution: $x_1 = 5/2$, $x_2 = 5/2$, $x_3 = 5/2$

Optimal Value: $Z = 15$.

Example 3

Solve the following LPP

Maximize $Z = 4x_1 + 3x_2$

Subject to –

$x_1 + x_2 \leq 50$

$x_1 + 2x_2 \geq 80$

$3x_1 + 2x_2 \geq 140$

$x_1, x_2 \geq 0$.

Solution

We introduce slack variables in "$\leq$" constraint and surplus and artificial variable in "$\geq$" type constraints to convert inequalities to equalities. Also we assign zero coefficient to slack and surplus variable and *–M* to artificial variable. Hence, the standard form of LPP is given as –

Maximize $Z = 4x_1 + 3x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$

Subject to–

$x_1 + x_2 + S_1 + 0S_2 + 0S_3 + 0A_1 + 0A_2 = 50$

$x_1 + 2x_2 + 0S_1 - S_2 + 0S_3 + A_1 + 0A_2 = 80$

$3x_1 + 2x_2 + 0S_1 + 0S_2 - S_3 + 0A_1 + A_2 = 140$

$x_1, x_2, S_1, S_2, S_3, A_1, A_2 \geq 0$.

The IBFS is obtained by considering the non-basic variables as zero i.e.,

$x_1 = 0,\ x_2 = 0\ , S_2 = 0, S_3 = 0$

Hence, the IBFS is

$$S_1 = 50,\ A_1 = 80,\ A_2 = 140$$

Matrix form of constraint equations is

$$AX = B$$

where,

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & -1 & 0 & 1 \end{pmatrix},\ X \begin{pmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ S_3 \\ A_1 \\ A_2 \end{pmatrix},\ B = \begin{pmatrix} S_1 \\ A_1 \\ A_2 \end{pmatrix}$$

With the IBFS, we move towards optimality using simplex table.

Simplex Table – 9

		$C_j \rightarrow$	4	3	0	0	0	–M	–M	
C_B	Basis	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	θ
0	S_1	50	1	1	1	0	0	0	0	50
–M	A_1	80	1	**2**	0	–1	0	1	0	80
–M	A_2	140	**3***	2	0	0	–1	0	1	**140/3** $\rightarrow$
	Z_j	–220M	–4M	–4M	0	M	M	–M	–M	
		$Z_j - C_j$	**–4 –4M** $\uparrow$	–3 –4M	0	M	M	0	0	

From the Simplex Table – 9, it is clear that, $Z_j - C_j$ is minimum for x_1 column. Hence, the variable x_1 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable A_2, indicating that A_2 should leave the basis.

Simplex Table – 10

		$C_j \rightarrow$	4	3	0	0	0	–M	
C_B	Basis	X_B	x_1	x_2	S_1	S_2	S_3	A_1	θ
0	S_1	10/3	0	**1/3***	1	0	1/3	0	**10** $\rightarrow$
–M	A_1	100/3	0	4/3	0	–1	1/3	1	25
4	x_1	140/3	1	2/3	0	0	–1/3	0	70
	Z_j	$\frac{560-100\text{M}}{3}$	4	$\frac{8-4\text{M}}{3}$	0	M	$\frac{-\text{M}-4}{3}$	–M	
		$Z_j - C_j$	0	$\frac{-4\text{M}}{3} - \frac{1}{3}$ $\uparrow$	0	M	$\frac{-\text{M}-4}{3}$	0	

From the Simplex Table – 10, it is clear that, $Z_j - C_j$ is minimum for x_2 column. Hence, the variable x_2 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_1, indicating that S_1 should leave the basis.

Simplex Table – 11

		$C_j \rightarrow$	4	3	0	0	0	–M
C_B	Basis	X_B	x_1	x_2	S_1	S_2	S_3	A_1
3	x_2	10	0	**1**	3	0	1	0
–M	A_1	20	0	0	–4	–1	–1	1
4	x_1	40	1	0	–2	0	–1	0
	Z_j	190 – 20M	4	3	1+4M	M	M–1	–M
		$Z_j - C_j$	0	0	1+4M	M	-1+M	0

Conclusion

From the Simplex Table – 11 it is observed that all $Z_j - C_j$ values are ≥ 0, the current solution is optimal. But artificial variable is present as basic variable (i.e., at non- zero level). Hence, the solution is said to be infeasible.

TWO-PHASE METHOD

The Two-Phase method is based on the following simple observation:

Suppose that you have a linear programming problem in canonical form and you wish to generate a feasible solution (not necessarily optimal) such that a given variable, say x_3, is equal to zero. Then, all you have to do is solve the linear programming problem obtained from the original problem by replacing the original objective function by x_3 and setting optimum value = minimum value.

If more than one variable is required to be equal to zero, then replace the original objective function by the sum of all the variables you want to set to zero.

Observe that, because of the non-negativity constraint, the sum of any collection of variables cannot be negative. Hence, the smallest possible feasible value of such a sum is zero. If the smallest feasible sum is strictly positive, then it is impossible to set all the designated variables to zero.

Two-Phase Simplex is designed to understand the process of solving linear programming problems which do not contain an identity sub-matrix in the initial tableau. Here, the artificial variables are introduced and the objective function is changed so that the Simplex Method will force the artificial variables out of the basis and replace them with decision variables. The two-phase procedure completes the study of the Simplex solution method and allows us to solve any LP problem to find its optimal solution or discover whether it is unbounded or infeasible.

Two-Phase Simplex method is an extended form of the Simplex Method. This is studied in two phases:

Phase I: Artificial variables are driven out of the basis.

Phase II: The original problem is solved to obtain optimum solution.

Since the solution of LPP is completed in two phases, it is called "Two-Phase Simplex Method".

This method is due to Dantzig, Orden and Wolfe.

ADVANTAGES OF TWO-PHASE OVER BIG-M METHOD

i. Although Big-M method is used to check the existence of a feasible solution, it may be computationally inconvenient because of the manipulation of the constant M. On the other hand, Two-phase method eliminates the constant M (very large value) from calculations.

ii. Another difficulty arises when the problem is to be solved on a digital computer. To use a digital computer, M must be assigned some numerical value which is much larger than the values $C_1, C_2, \ldots$ in the objective function. But, a computer has only a fixed number of digits. On the other hand, Two-phase method assigns cost coefficient (–1) for artificial variables.

TWO-PHASE SIMPLEX ALGORITHM

The Two Phase Simplex method is used to solve a given problem in which some artificial variables are involved. The solution obtained in two phases is as follows:

Phase-I

In this phase, the Simplex method is applied to a specially constructed auxiliary LPP leading to a final Simplex table containing a basic feasible solution to the original problem.

Step 1: Assign a cost – 1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function.

Step 2: Construct the auxiliary LPP in which the new objective function Z^* is to be maximized subject to the given constraints.

Step 3: Solve the auxiliary problem by Simplex method until one of the following three possibilities arise:

i. $\text{Max}\, Z^* < 0$ i.e. $\Delta_j = (Z_j - C_j) < 0$ and at least one artificial variable appears in the optimum basis at negative level. In this case, the problem does not possess any feasible solution.

ii. $\text{Max}\, Z^* > 0$ i.e. $\Delta_j > 0$ and at least one artificial variable appears in the optimum basis at zero level. In this case, proceed to Phase-II.

iii. $\text{Max}\, Z^* = 0$ and no artificial variable appears in the optimum basis. In this case also proceed to Phase-II.

Phase-II

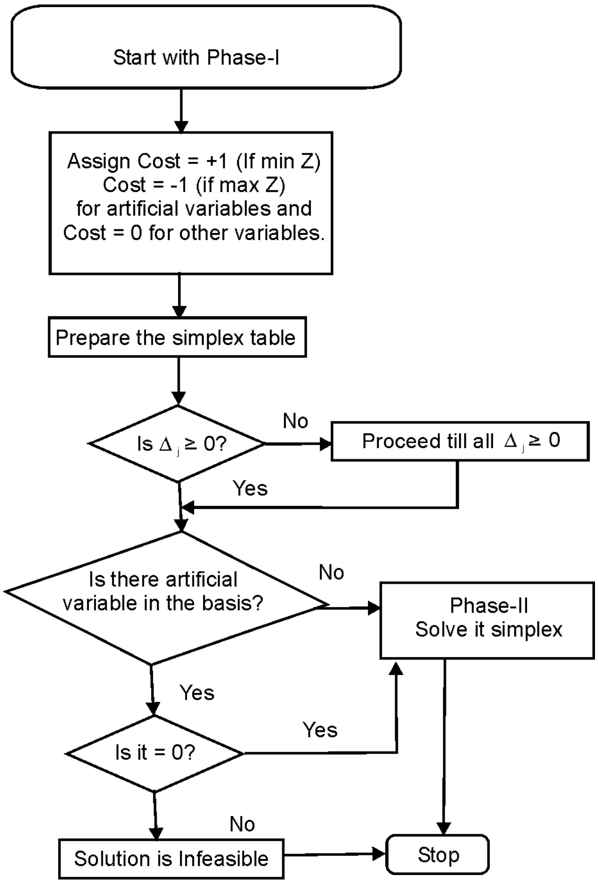
Now assign the actual costs to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by Simplex method subject to the given constraints. In other words, Simplex method is applied to the modified Simplex table obtained at the end of Phase-I, until an optimum feasible solution (if exists) has been attained. The artificial variables which are non-basic at the end of Phase-I are removed.

FLOW CHART

The flow chart of algorithm for two phase simplex method can be depicted as follows:

To Correct

Figure



Example 4

Use Two-Phase Simplex method to solve the following LPP

Maximize $Z = 5x_1 + 3x_2$

Subject to

$$2x_1 + x_2 \leq 1$$

$$x_1 + 4x_2 \geq 6$$

$$x_1, x_2 \geq 0.$$

Solution

Phase-I

To convert inequations into equations we introduce slack, surplus and artificial variables in the above constraints.

$$2x_1 + x_2 + S_1 + 0S_2 + 0A_1 = 1$$

$$x_1 + 4x_2 + 0S_1 - S_2 + A_1 = 6.$$

We assign cost coefficient '–1'to the artificial variable and '0' to all other variables, in the objective function. The LPP is given as –

Maximize $Z = 0x_1 + 0x_2 + 0S_1 + 0S_2 - A_1$

Subject to

$$2x_1 + x_2 + S_1 + 0S_2 + 0A_1 = 1$$

$$x_1 + 4x_2 + 0S_1 - S_2 + A_1 = 6$$

$$x_1, x_2, S_1, S_2, A_1 \geq 0.$$

The IBFS is obtained by setting the non-basic variables to zero i.e.,

$$x_1 = 0,\ x_2 = 0$$

Hence, the IBFS is

$$S_1 = 1,\ A_1 = 6,\ S_2 = 0$$

Matrix form of constraint equations is

$$AX = B$$

where,

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 4 & 0 & -1 & 1 \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ A_1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 6 \end{pmatrix}.$$

Simplex Table – 12

		$C_j \rightarrow$	0	0	0	0	– 1	
C_B	Basis	X_B	x_1	x_2	S_1	S_2	A_1	θ
0	S_1	1	2	**1***	1	0	0	**1** $\rightarrow$
– 1	A_1	6	1	4	0	–1	1	3/2
	Z_j	– 6	–1	–4	0	1	–1	
		$Z_j - C_j$	–1	–4 $\uparrow$	0	1	0	

From the Simplex Table – 12, it is clear that, $Z_j - C_j$ is minimum for x_2 column. Hence, the variable x_2 should enter the basis.

Again from θ-rule the ratio is minimum for the variable S_1, indicating that S_1 should leave the basis.

Simplex Table – 13

		$C_j \rightarrow$	0	0	0	0	– 1
C_B	Basis	X_B	x_1	x_2	S_1	S_2	A_1
0	x_2	1	2	1	1	0	0
–1	A_1	2	–7	0	– 4	–1	1
	Z_j	–2	7	0	4	1	–1
		$Z_j - C_j$	7	0	4	1	0

Since $Z_j - C_j$ row contains all ≥ 0, optimum solution has been attained. But since, artificial variable A_1 appears as basic variable, the above problem does not possess any feasible solution.

The given LPP has infeasible solution. Hence, we will not proceed to Phase II.

Example 5

A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 calories. Two foods *A* and *B* are available at a cost of Rs.4 and Rs.3 per unit respectively. If one unit of Food *A* contains 200 units of vitamins, 1 unit of minerals and 40 calories and one unit of food *B* contains 100 units of vitamins, 2 units of minerals and 40 calories, find by Two-Phase method, what combination of food be used to have the least cost?

Solution

The given information can be presented in a tabular form as follows:

	Vitamins	Minerals	Calories	Cost per unit
Food *A*	200	1	40	4
Food *B*	100	2	40	3
Minimum requirement	4000	50	1400	

The mathematical form of LPP with decision variables x_1, number of units of Food *A* and x_2, number of units of Food *B*, is given as –

Minimize $Z = 4x_1 + 3x_2$

Subject to

$200x_1 + 100x_2 \geq 4000$ or $2x_1 + 1x_2 \geq 40$

$x_1 + 2x_2 \geq 50$

$40x_1 + 40x_2 \geq 1400$ or $x_1 + x_2 \geq 35$

$x_1, x_2 \geq 0.$

Phase-I

To convert in equations into equations we introduce surplus and artificial variables in the above constraints.

$2x_1 + x_2 - S_1 + A_1 = 40$

$x_1 + 2x_2 - S_2 + A_2 = 50$

$x_1 + x_2 - S_3 + A_3 = 35.$

We assign cost coefficient '+1' to the artificial variable (since the problem is of minimization) and '0' to all other variables, in the objective function. The LPP is given as –

Minimize $Z = 0x_1 + 0x_2 + 0S_1 + 0S_2 + 0S_3 + A_1 + A_2 + A_3$

Subject to

$2x_1 + x_2 - S_1 + A_1 = 40$

$x_1 + 2x_2 - S_2 + A_2 = 50$

$x_1 + x_2 - S_3 + A_3 = 35$

$x_1, x_2, S_1, S_2, S_3, A_1, A_2, A_3 \geq 0$

The IBFS is obtained by considering the non-basic variables as zero i.e.,

$x_1 = 0,\ x_2 = 0$

Hence, the IBFS is

$$A_1 = 40, A_2 = 50, A_3 = 35$$

Matrix form of constraint equations is

$$AX = B$$

where,

$$A = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}, \ X = \begin{pmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ S_3 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}, \ B = \begin{pmatrix} 40 \\ 50 \\ 35 \end{pmatrix}.$$

Simplex Table – 14

		$C_j \rightarrow$	0	0	0	0	0	1	1	1	
C_B	Basis	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3	θ
1	A_1	40	2*	1	–1	0	0	1	0	0	20 $\rightarrow$
1	A_2	50	1	2	0	–1	0	0	1	0	50
1	A_3	35	1	1	0	0	–1	0	0	1	35
	Z_j	125	4	4	–1	–1	–1	1	1	1	
		$Z_j - C_j$	4 $\uparrow$	4	–1	–1	–1	0	0	0	

From the Simplex Table – 14, it is clear that, the maximum $Z_j - C_j$ (since the problem is of minimization) is for x_1 column and x_2 column. Arbitrarily, we select x_1 column. Hence, the variable x_1 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable A_1, indicating that A_1 should leave the basis.

Simplex Table – 15

		$C_j \rightarrow$	0	0	0	0	0	1	1	
C_B	Basis	X_B	x_1	x_2	S_1	S_2	S_3	A_2	A_3	θ
0	x_1	20	1	1/2	–1/2	0	0	0	0	40
1	A_2	30	0	3/2*	1/2	–1	0	1	0	20 $\rightarrow$
1	A_3	15	0	1/2	1/2	0	–1	0	1	30
	Z_j	45	0	2	1	–1	–1	1	1	
		$Z_j - C_j$	0	2 $\uparrow$	1	–1	–1	0	0	

From the Simplex Table – 15, it is clear that, the maximum $Z_j - C_j$ is for x_2 column. Hence, the variable x_2 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable A_2, indicating that A_2 should leave the basis.

Simplex Table – 16

		$C_j \rightarrow$	0	0	0	0	0	1	
C_B	Basis	X_B	x_1	x_2	S_1	S_2	S_3	A_3	θ
0	x_1	10	1	0	–2/3	1/3	0	0	–ve
0	x_2	20	0	1	1/3	–2/3	0	0	60
1	A_3	5	0	0	1/3*	1/3	–1	1	15 $\rightarrow$
	Z_j	5	0	0	1/3	1/3	–1	1	
		$Z_j - C_j$	0	0	1/3 $\uparrow$	1/3	1	0	

From the Simplex Table – 16, it is clear that, the maximum $Z_j - C_j$ is for S_1 column and S_2 column. Arbitrarily, we select S_1 column. Hence, the variable S_1 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable A_3, indicating that A_3 should leave the basis.

Simplex Table – 17

		$C_j \rightarrow$	0	0	0	0	0
C_B	Basis	X_B	x_1	x_2	S_1	S_2	S_3
0	x_1	20	1	0	0	1	–2
0	x_2	15	0	1	0	–1	1
0	S_1	15	0	0	1	1	–3
	Z_j	0	0	0	0	0	0
		$Z_j - C_j$	0	0	0	0	0

Since all $Z_j - C_j$ values are zero, we can enter into Phase-II.

Phase-II

Simplex Table – 18

In this phase, original costs of variables are considered.

		$C_j \rightarrow$	4	3	0	0	0	
C_B	Basis	X_B	x_1	x_2	S_1	S_2	S_3	θ
4	x_1	20	1	0	0	1	–2	20
3	x_2	15	0	1	0	–1	1	–ve
0	S_1	15	0	0	1	1*	–3	15 $\rightarrow$
	Z_j	125	4	3	0	1	–5	
		$Z_j - C_j$	0	0	0	1 $\uparrow$	–5	

From the Simplex Table – 18, it is clear that, most negative $Z_j - C_j$ is for S_2 column. Hence, the variable S_2 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_1, indicating that S_1 should leave the basis

Simplex Table – 19

		$C_j \rightarrow$	4	3	0	0	0
C_B	Basis	X_B	x_1	x_2	S_1	S_2	S_3
4	x_1	5	1	0	–1	0	1
3	x_2	30	0	1	1	0	– 2
0	S_2	15	0	0	1	1	– 3
	Z_j	110	4	3	–1	0	– 2
		$Z_j - C_j$	0	0	–1	0	– 2

Conclusion

In Simplex Table – 19, we observe that all $Z_j - C_j$ are ≤ 0 (for minimization of objective function). Hence, we are with optimum solution.

Optimum Solution: $x_1 = 5$, $x_2 = 30$.

Optimal Value: Minimum of $Z = 110$.

Hence, the diet for a sick person must contain 5 units of food *A* and 30 units of food *B* so that, the cost is minimum.

SUMMARY

- An Artificial Variable is a non-negative quantity added to the left side of a constraint of an LP problem in standard form to obtain the basis matrix.
- The Big-M method is a method of solving a linear programming problem involving artificial variables. In this method, we assign a very high penalty (say M) to the artificial variable in the objective function.
- Two-Phase Simplex is designed to understand the process of solving linear programming problems which do not contain an identity sub-matrix in the initial tableau, i.e., the constraints are of '=' or '≥' type.
- Two-phase method is an extended form of the Simplex Method.
- Two-phase method is studied in two phases: (a) Phase-I: Artificial variables are thrown out of the basis. (b) Phase-II: Optimum solution is attained.

Exercise

1. What do you mean by the Two-phase method for solving LPP? Why is it used?
2. What are artificial variables? Why do we need them? Describe the two-phase method of solving LPP with artificial variables.
3. Explain the steps involved in solving LPP using two-phase method.
4. In Phase-I, if the LP is of maximization type, do we maximize the sum of the artificial variables in Phase-I? Explain.
5. A firm has an advertising budget of Rs.7,20,000. It wishes to allocate this budget to two media: magazines and televisions, so that total exposure is maximized. Each page of the magazine advertising is estimated to result in 60,000 exposures, whereas each spot on television is estimated to result in 1,20,000 exposures. Each page of the magazine advertising costs Rs.9,000 and each spot on television costs Rs.12,000. An additional condition that the firm has specified is that atleast two pages of the magazine advertising and at least 3 spots on television be used. Determine the optimum media-mix for this firm.
6. A manufacturer produces two different models x and y of the same product. The raw materials A and B are required for production. Atleast 18 kg of A and 12 kgs of B must be used daily. Also, at most 34 hours of labor are to be utilized; 2 kg of A are needed for each model x and 1 kg of A for each model y. For each model x and y, 1 kg of B is required. It takes 3 hours to manufacture a model x and 2 hours to manufacture a model y. The profit is Rs.50 for each model x and Rs.50 for each model y. How many units of each model should be produced to maximize the profit? Use the two-phase simplex method.
7. The manager of an oil refinery must decide on the optimal mix of two possible blending processes, of which the input and output per production run are as follows:

Process	Input units		Output units	
	Crude A	Crude B	Gasoline X	Gasoline Y
1	5	3	5	8
2	4	5	4	4

The maximum amounts available of crudes A and B is 200 units and 150 units respectively. Market requirements show that atleast 100 units of gasoline X and 80 units of gasoline Y must be produced. The profits per production run from process I and process II are Rs.300 and Rs.400 respectively. Solve the LPP by the two-phase simplex method.

8. An animal feed company must produce 200 kg of a mixture consisting of ingredients x_1 and x_2, daily. x_1 costs Rs.3 per kg. and x_2 costs Rs.8 per kg. No more than 80 kg of x_1 can be used and atleast 60 kgs. of x_2 must be used. Find how much of each ingredient should be used if the company wants to minimize cost. Use the two-phase simplex method.
9. A rubber company is engaged in the production of three different types of tyres. These tyres are produced at company with two different capacities. In a normal 8 hours day, plant I produces 100, 200 and 200 tyres of type A, B and C respectively. Plant II produces 120, 120 and 400 tyres of type A, B and C respectively. The monthly demand for A, B and C is 5,000, 6,000 and 14,000 tyres respectively. The daily cost of operation of plant I and II is Rs.5,000 and Rs. 7,000 respectively. Find the minimum number of days of operation per month at two different plants to minimize the total cost while meeting the demand.

Solve the following problems by the two-phase simplex method:

10. Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 2$

$x_2 \leq 4$

$x_1, x_2 \geq 0.$

11. Max $Z = 5x_1 + 8x_2$

Subject to

$3x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$x_1 + x_2 \leq 5$

$x_1, x_2 \geq 0.$

12. Max $Z = x_1 + 1.5x_2 + 2x_3 + 5x_4$

Subject to

$3x_1 + 2x_2 + 4x_3 + x_4 \leq 6$

$2x_1 + x_2 + x_3 + 5x_4 \leq 4$

$2x_1 + 6x_2 - 8x_3 + 4x_4 = 0$

$x_1 + 3x_2 - 4x_3 + 3x_4 = 0$

$x_1, x_2 \geq 0.$

13. Min $Z = x_1 - 2x_2 - 3x_3$

Subject to

$-2x_1 + x_2 + 3x_3 = 2$

$2x_1 + 3x_2 + 4x_3 = 1$

$x_j \geq 0 \ (j = 1, 2, 3).$

14. Max $Z = 3x_1 + 2x_2 + x_3 + 4x_4$

Subject to

$4x_1 + 5x_2 + x_3 - 3x_4 = 5$

$2x_1 - 3x_2 - 4x_3 + 5x_4 = 7$

$x_1 + x_2 + 2.5x_3 - 4x_4 = 6$

$x_1, x_2, x_3, x_4 \geq 0.$

15. Max $Z = 5x_1 - 2x_2 + 3x_3$

Subject to

$2x_1 + 2x_2 - x_3 \geq 2$

$3x_1 - 4x_2 \leq 3$

$x_2 + 3x_3 \leq 5$

$x_1, x_2, x_3 \geq 0.$

16. Max $Z = 2x_1 + 3x_2 + 5x_3$

Subject to

$3x_1 + 10x_2 + 5x_3 \le 15$

$x_1 + 2x_2 + x_3 \ge 4$

$33x_1 - 10x_2 + 9x_3 \le 33$

$x_1, x_2, x_3 \ge 0.$

17. Max $Z = 500x_1 + 1400x_2 + 900x_3$

Subject to

$x_1 + x_2 + x_3 = 100$

$12x_1 + 35x_2 + 15x_3 \ge 25$

$8x_1 + 3x_2 + 4x_3 \ge 6$

$x_1, x_2, x_3 \ge 0.$

18. Min $Z = 200x_1 + 400x_2$

Subject to

$x_1 + x_2 \ge 200$

$x_1 + 3x_2 \ge 400$

$x_1 + 2x_2 \le 350$

$x_1, x_2 \ge 0.$

19. Max $Z = 3x_1 + 2x_2 + 2x_3$

Subject to

$5x_1 + 7x_2 + 4x_3 \ge 7$

$-4x_1 + 7x_2 + 5x_3 \ge -2$

$3x_1 + 4x_2 - 6x_3 \ge 29/7$

$x_1, x_2, x_3 \ge 0.$

20. Explain the Big-M method of Simplex method to solve a LPP.
21. What are the advantages of Two- Phase method over Big-M method?
22. Distinguish between Big-M Method and Two-Phase Method.
23. Maximize $Z = x_1 - x_2 + x_3 + x_4 + x_5 - x_6$

Subject to

$x_1 + x_4 + 6x_6 = 9$

$3x_1 + x_2 - 4x_3 + 2x_6 = 2$

$x_1 + 2x_3 + x_5 + 2x_6 = 6$

$x_i \ge 0 \ (i = 1,2,3,4,5,6).$

24. Min $Z = 4x_1 + 8x_2 + 3x_3$

Subject to

$x_1 + x_2 \ge 20$

$2x_1 + x_3 \ge 5$

$x_1, x_2, x_3 \ge 0.$

25. Max $Z = 3x_1 + 2.5x_2$

Subject to

$2x_1 + 4x_2 \geq 40$

$3x_1 + 2x_2 \geq 50$

$x_1, x_2 \geq 0.$

26. Min $Z = 2x_1 + 3x_2$

Subject to

$x_1 + x_2 \geq 5$

$x_1 + 2x_2 \geq 6$

$x_1, x_2 \geq 0.$

27. Min $Z = 3x_1 + 2x_2 + x_3$

Subject to

$2x_1 + 5x_2 + x_3 = 12$

$3x_1 + 4x_2 = 11$

x_1 is unrestricted, $x_2, x_3 \geq 0.$

Chapter V

Duality in LPP

After reading this chapter, you will be conversant with:

- Economic Interpretation of Duality
- Formulation of the General Dual Problem
- Primal Dual Relationship
- Symmetric and Asymmetric Forms of LPP
- Rules for Constructing the Dual Problems
- Dual of Dual Problem
- Statements of Some Important Theorems

Introduction

Every Linear Programming Problem (called primal) is associated with another Linear Programming Problem (called dual). Either of the problem can be considered as primal with the other one as its dual. The optimal solution for the primal and dual are equivalent but they are derived through alternative procedures. The dual contains economic information useful to management and it may also be easier to solve, than a primal problem. Generally, LPP primal involves maximizing a profit function subject to the limited resource constraints, the dual will involve minimizing total opportunity cost subject to the unlimited profit constraints.

ECONOMIC INTERPRETATION OF DUALITY

The economic interpretation of duality is discussed with the help of the following example:

Let the following table give the amounts of two vitamins V_1 and V_2 per unit, present in two different foods F_1 and F_2 respectively:

Vitamin	Food		Daily requirement
	F_1	F_2	
V_1	8	10	110
V_2	7	12	130
Cost per unit	12	20	

The last column of the above table represents the number of units of the minimum daily requirement for the two vitamins, whereas the last row represents the cost per unit of the two foods.

Formulating the above Problem as Primal LPP

It is a customer's problem (in cost point of view). The problem is to determine the minimum quantities of the two foods F_1 and F_2 so that the minimum daily requirement of the two vitamins is met and that, at the same time, the cost of purchasing these quantities of F_1 and F_2 is minimum.

Decision variables

x_1 = Number of units of food F_1

x_2 = Number of units of food F_2.

Constraints

Based on minimum daily requirements.

$$8x_1 + 10x_2 \geq 110$$

$$7x_1 + 12x_2 \geq 130.$$

Non-negativity restriction

$$x_1 \geq 0, x_2 \geq 0.$$

Objective function

Minimize the total cost of purchasing food F_1 and F_2

i.e. $12x_1 + 20x_2$

The above primal can also be represented as:

Minimize

$$Z = 12x_1 + 20x_2$$

Subject to the constraints

$$8x_1 + 10x_2 \geq 110$$

$$7x_1 + 12x_2 \geq 130$$

$$x_1 \geq 0, x_2 \geq 0.$$

Formulating the above Problem as Dual LPP

It is a wholesale dealer's problem (in quantity point of view). The problem is to fix the maximum per unit selling prices for the two vitamins V_1 and V_2 in such a way that the resulting prices of foods F_1 and F_2 do not exceed their market prices.

The local shopkeepers purchase the vitamins from him and form the foods F_1 and F_2 (with combination of vitamins as given in the above table). The dealer knows very well that the foods have their market value only because of their vitamin contents.

Decision variables

y_1 = Cost per unit of vitamin V_1

y_2 = Cost per unit of vitamin V_2.

Constraints

Based on minimum cost per unit of vitamins.

$$8y_1 + 7y_2 \leq 12$$

$$10y_1 + 12y_2 \leq 20.$$

Non-negativity restriction

$$y_1 \geq 0, y_2 \geq 0.$$

Objective function

Maximize the total profit on food F_1 and F_2

i.e. $110y_1 + 130y_2$

The above dual in mathematical form can be represented as

Maximize

$$Z^* = 110y_1 + 130y_2$$

Subject to the constraints

$$8y_1 + 7y_2 \leq 12$$

$$10y_1 + 12y_2 \leq 20$$

$$y_1 \geq 0, y_2 \geq 0.$$

FORMULATION OF THE GENERAL DUAL PROBLEM

PRIMAL

In order to understand the formulation of dual, we will define the dual problem when its primal problem is given in the following form:

Maximize $z = c_1x_1 + c_2x_2 + ... + c_nx_n$

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1j}x_j + ... + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2j}x_j + ... + a_{2n}x_n \leq b_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n \leq b_i$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mj}x_j + ... + a_{mn}x_n \leq b_m \text{ and}$$

$$(x_1, x_2, ..., x_n) \geq 0.$$

DUAL

The dual of this problem is expressed as:

$$\text{Minimize } z^* = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

Subject to the constraints

$$a_{11} y_1 + a_{21} y_2 + \ldots + a_{i1} y_i + \ldots + a_{m1} y_m \geq c_1$$

$$a_{12} y_1 + a_{22} y_2 + \ldots + a_{i2} y_i + \ldots + a_{m2} y_m \geq c_2$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$a_{1j} y_1 + a_{2j} y_2 + \ldots + a_{ij} y_i + \ldots + a_{mj} y_m \geq c_j$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{in} y_i + \ldots + a_{mn} y_m \geq c_n$$

and $y_1, y_2, \ldots, y_m \geq 0$

where, $y_1, y_2, \ldots, y_m$ are dual decision variables.

In simple words,

The Dual of the LPP is obtained by,

i. Transposing the coefficient matrix.

ii. Interchanging the role of constant terms on RHS of constraints and the coefficients of the objective function.

iii. Reverting the inequalities.

iv. Minimizing the objective function instead of maximizing it.

PRIMAL DUAL RELATIONSHIP

The primal-dual construction relationship can be summarized as follows:

		Primal Variables x_1 $x_2 \ldots\ldots\ldots x_n$	Minimize Z^*
Dual Variables	y_1	a_{11} $a_{122} \ldots\ldots a_{1n}$	$\leq b_1$
	y_2	a_{21} $a_{22} \ldots\ldots a_{2n}$	$\leq b_2$
	$\vdots$	$\vdots$ $\vdots \ldots\ldots \vdots$	$\vdots$
	$\vdots$	$\vdots$ $\vdots \ldots\ldots \vdots$	$\vdots$
	y_m	a_{m1} $a_{m2} \ldots\ldots a_{mn}$	$\leq b_m$
Maximize Z		$\geq c_1$ $\geq c_2 \ldots\ldots \geq c_n$	

Primal vs. Dual

S. No.	Standard Primal Problem	Dual Problem
1.	Objective function: Maximization type	Objective function: Minimization type
2.	Requirement vector i.e., b	Price Vector i.e., C
3.	Coefficient matrix i.e., A	Transpose of Coefficient matrix i.e., A^T
4.	Constraints with '$\leq$' sign	Constraints with '$\geq$' sign.
5.	ith inequality	ith variable
6.	ith constraint inequality	ith variable restricted i.e., $y_i \geq 0$
7.	Variable	Relation
8.	If ith variable $x_i > 0$.	ith relation a strict in equality
9.	ith variable x_i unrestricted in sign	ith constraint a strict in equality
10.	If ith slack variable positive	ith variable zero
11.	If ith variable zero	ith surplus variable positive
12.	Finite optimum solution	Finite optimum solution with equal optimal value of objective function
13.	Unbounded solution	No solution or an Unbounded solution

INTERPRETING THE PRIMAL DUAL RELATIONSHIP

For interpreting the optimal solution to the primal (or dual) its solution values can be read directly from the optimal solution table of the dual (or primal). It can be summarized in the following steps:

Step 1: Locate the slack/surplus variables in the dual (or primal) problem. These variables correspond to the primal (or dual) basic variable in the optimal solution.

Step 2: The values of net evaluations in the dual solution corresponding to the columns of the slack/surplus variables without sign gives directly the optimal values of the primal basic variables.

Step 3: Value for slack/surplus variables of the primal are given by the net evaluation corresponding to the non-basic variables of the dual solution with change in sign.

Step 4: The value of the objective function will be same for both primal and dual problem.

SYMMETRIC AND ASYMMETRIC FORMS OF LPP

Symmetric Forms of LPP

There are two types of symmetric forms of LPP:

TYPE I

An LPP is said to be in the symmetric form, if all the constraints are in equations of $\leq$ type. i.e., the constraints are $AX \leq b$, $b \in R^m$ where, A is an $m \times n$ real matrix.

For example,

$$2x_1 + 3x_2 \leq 5$$

$$x_1 + 2x_2 \leq 1.$$

TYPE II

An LPP is said to be in the symmetric form, if all the constraints are in equations of $\geq$ type, i.e., the constraints are $AX \geq b,\ b \in R^m$ where, A is an $m \times n$ real matrix.

For example,

$$2x_1 + 3x_2 \geq 5$$

$$x_1 + 2x_2 \geq 1.$$

Asymmetric Forms of LPP

An LPP is said to be in the asymmetric form, if all the constraints are inequations of "$\leq$", "$\geq$"and/ or "=" type, i.e., the constraints are $AX \geq \text{or} \leq \text{or} = b,\quad b \in R^m$ where, A is an $m \times n$ real matrix.

For example,

i. $2x_1 + 3x_2 \geq 5$

$x_1 + 2x_2 \leq 1$

ii. $2x_1 + 3x_2 \geq 5$

$x_1 + 2x_2 = 1.$

Matrix Form of Primal Problem

Maximize

$$Z = CX,\ C \in R^n$$

Subject to the constraints $AX \leq b,\ b \in R^m$ where, A is an $m \times n$ real matrix. $X \geq 0$.

Matrix Form of Dual Problem

Minimize $Z^* = b^T Y$

Subject to the constraints $A^T Y \geq C^T,\ \ C \in R^n \quad Y \geq 0$.

Here, the dual variables are unrestricted in sign.

RULES FOR CONSTRUCTING THE DUAL PROBLEMS

i. If the objective of one problem is to maximize, the objective of the other is to minimize.

ii. The maximization problem should have all $\leq$ constraints and minimization problems should have $\geq$ constraints. If in a maximization case any constraint is $\geq$ it can be multiplied by –1 and signs can be got reversed. Similarly, if in a minimization case any problem has got $\leq$ type, then it can be reversed by multiplying both sides by –1.

iii. The number of primal decision variables (x_i) equals the number of dual constraints and the number of primal constraints equals the number of dual variables (y_i) i.e., each dual variable corresponds to one constraint in the primal and vice-versa. Therefore, if the primal problem has 'm' constraints and 'n' variables then dual problem will have 'n' constraints and 'm' variables. Such condition is usually violated in case of = type constraints, and should be resolved before proceeding to the solution.

iv. If the inequalities are $\leq$ type, then in the dual problem we have $\geq$ type and vice-versa.

v. The c_j of the primal in the objective function appears as RHS constants of the dual constraints, and the primal RHS constants (b_i) appears as unit contribution rate, i.e., c_j of the dual decision variable in the objective function. The matrix of constraint coefficient for one problem is the transpose of the matrix of constraints coefficient of the other problem, i.e., rows in the primal becomes columns in the dual.

Advantages of Duality

The knowledge of the dual is important for two main reasons. They are:

i. Obtaining solution through dual is easier than through primal.

ii. Duality is not only restricted to LPP but frequently occurs in Economics, Physics, Engineering, Mathematics and other fields.

Applications

It is applied in Economics, Physics and Mathematics and various other fields:

Economics: It is used in the formulation of the input and output systems. The economic interpretation of the dual is found useful in making future decisions in the activities.

Physics: It is used in the parallel circuit and series circuit theory.

Mathematics: In game theory, it is used in finding optimal strategies of a player when he minimizes his losses.

DUAL OF DUAL PROBLEM

The dual of the dual LPP is nothing but its primal.

Proof

Primal

Maximize $z = c_1x_1 + c_2x_2 + ... + c_nx_n$

Subject to the constraints

$$\left.\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + ... + a_{1j}x_j + ... + a_{1n}x_n \le b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2j}x_j + ... + a_{2n}x_n \le b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n \le b_i \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mj}x_j + ... + a_{mn}x_n \le b_m \\
\text{and } x_1, x_2, ..., x_n \ge 0.
\end{array}\right\} \qquad \text{… (1)}$$

Dual

Minimize $z^* = b_1y_1 + b_2y_2 + ... + b_my_m$

Subject to the constraints

$$\left.\begin{array}{l}
a_{11}y_1 + a_{21}y_2 + ... + a_{i1}y_i + ... + a_{m1}y_m \ge c_1 \\
a_{12}y_1 + a_{22}y_2 + ... + a_{i2}y_i + ... + a_{m2}y_m \ge c_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
a_{1j}y_1 + a_{2j}y_2 + ... + a_{ij}y_i + ... + a_{mj}y_m \ge c_j \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
a_{1n}y_1 + a_{2n}y_2 + ... + a_{in}y_i + ... + a_{mn}y_m \ge c_n \\
\text{and } y_1, y_2, ..., y_m \ge 0.
\end{array}\right\} \qquad \text{… (2)}$$

Now, our aim is to construct the dual of dual (2). For this, we shall first express the above dual (2) in the standard form of the primal by multiplying through by –1. Thus, we obtain,

Max $-Z^* = -b_1y_1 - b_2y_2 - \ldots - \ldots - b_my_m$

Subject to the constraints

$$\left.\begin{array}{l} -a_{11}y_1 - a_{21}y_2 - \ldots - a_{i1}y_i - \ldots - a_{m1}y_m \le -c_1 \\ -a_{12}y_1 - a_{22}y_2 - \ldots - a_{i2}y_i - \ldots - a_{m2}y_m \le -c_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ -a_{1j}y_1 - a_{2j}y_2 - \ldots - a_{ij}y_i - \ldots - a_{mj}y_m \le -c_j \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ -a_{1n}y_1 - a_{2n}y_2 - \ldots - a_{in}y_i - \ldots - a_{mn}y_m \le -c_n \\ \text{and } y_1, y_2, \ldots, y_m \ge 0. \end{array}\right\} \quad \ldots (3)$$

***Dual of Dual*:** Now, constructing the dual of dual (2) i.e., for (3) according to the definition, we get,

Minimize $z_v = -c_1v_1 - c_2v_2 - \ldots - c_nv_n$

Subject to the constraints

$$\left.\begin{array}{l} -a_{11}v_1 - a_{12}v_2 - \ldots - a_{1j}v_j - \ldots - a_{1n}v_n \ge -b_1 \\ -a_{21}v_1 - a_{22}v_2 - \ldots - a_{2j}v_j - \ldots - a_{2n}v_n \ge -b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ -a_{i1}v_1 - a_{i2}v_2 - \ldots - a_{ij}v_j - \ldots - a_{in}v_n \ge -b_i \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ -a_{m1}v_1 - a_{m2}v_2 - \ldots - a_{mj}v_j - \ldots - a_{mn}v_n \ge -b_m \end{array}\right\} \quad \ldots (4)$$

and $v_1, v_2, \ldots, v_n \ge 0.$

Now, multiplying (4) through by –1, we get the equivalent system:

Maximize $z'_v = c_1v_1 + c_2v_2 + \ldots + c_nv_n$, $(z'_v = -z_v)$

Subject to the constraints

$$\left.\begin{array}{l} a_{11}v_1 + a_{12}v_2 + \ldots + a_{1j}v_j + \ldots + a_{1n}v_n \le b_1 \\ a_{21}v_1 + a_{22}v_2 + \ldots + a_{2j}v_j + \ldots + a_{2n}v_n \le b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{i1}v_1 + a_{i2}v_2 + \ldots + a_{ij}v_j + \ldots + a_{in}v_n \le b_i \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \ldots + a_{mj}v_j + \ldots + a_{mn}v_n \le b_m \end{array}\right\} \quad \ldots (5)$$

and $v_1, v_2, \ldots, v_n \ge 0$

which is the system identical to the primal (1).

Hence, dual of the dual is the primal.

Theorem

If X is any feasible solution to the primal problem and Y is any feasible solution to the dual problem, then $CX \le b^T Y$, i.e., $z_x \le z_y$ or given problems are:

$$AX \le b,\ X \ge 0,\ \max z_x = CX \qquad \dots (6)$$

$$A^T W \ge C^T, W \ge 0,\ \min z_w = b^T Y \qquad \dots (7)$$

Then, for any feasible solution X for (6) and any feasible solution W for (7), $CX \le b^T W$, i.e., $z_x \le z_y$.

Proof

Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_m)$ be any feasible solutions to the primal and dual problems, respectively. We have,

$$CX = \sum_{j=1}^{n} c_j x_j \le \sum_{j=1}^{n} \left[\sum_{i=1}^{m} a_{ij} y_i \right] x_j$$

$$\left[\text{Since } c_j \le \sum_{i=1}^{m} a_{ij} y_i \text{ is the } j\text{th constraint of the dual (2)} \right]$$

$$= \sum_{i=1}^{m} y_i \left[\sum_{i=1}^{m} a_{ij} x_j \right]$$

$$= \sum_{i=1}^{m} y_i b_i = b^T Y$$

$$\left[\text{Since } \sum_{j=1}^{n} a_{ij} x_j \le b_i \text{ is the } i\text{th constraint of the primal (1)} \right]$$

$$CX \le b^T Y, \text{or, } z_x \le z_y$$

Hence, the proof.

STATEMENTS OF SOME IMPORTANT THEOREMS

Fundamental Theorem of Duality

Statement

a. If primal (or the dual) problem has a finite optimal solution, then the dual (or primal) problem also has a finite optimal solution.

   Furthermore, the optimal values of the objective function in both the problems are the same, i.e., $\text{Max}\, z_x = \text{Min}\, z_y$.

b. If primal (or dual) problem has an **unbounded** optimum solution, then the dual (or primal) problem has **no feasible** solution at all.

c. Both problems may be infeasible, i.e., may not have any solution.

Existence Theorem

Statement: If there does not exist any finite optimum solution to the primal (dual) problem, then there does not exist any feasible solution to the dual (primal) problem.

Complementary Slackness Theorem

Statement: At optimality, if the primal constraint holds as a strict inequality, then the complementary dual constraint holds as an equality. Conversely, if a dual constraint holds as a strict inequality, its complementary primal constraint holds as an equality.

Theorem

If any variable of the primal problem is unrestricted in sign, the corresponding constraint in the dual will be a strict equality.

Theorem

If the *k*th constraint of the primal problem is an equality, then the dual variable is unrestricted in sign.

Example 1

Show that Dual of Dual is Primal for the following LPP:

Min $z = 50x_1 + 120x_2$

Subject to the constraints

$2x_1 + 4x_2 \le 80$

$3x_1 + x_2 \le 60$

and $x_1, x_2 \ge 0.$

Solution

The dual of the above primal is in symmetric form.

Max $z = 80y_1 + 60y_2$

Subject to the constraints

$2y_1 + 3y_2 \ge 50$

$4y_1 + y_2 \ge 120$

and $y_1, y_2 \ge 0.$

The dual of the dual is given as –

Max $z = 50x_1 + 120x_2$

Subject to the constraints

$2x_1 + 4x_2 \le 80$

$3x_1 + x_2 \le 60$

and $x_1, x_2 \ge 0$

which is the primal.

Hence, shown.

Example 2

Write the Dual of the following LPP:

Min $z = 5x_1 - 6x_2 + 4x_3$

Subject to the constraints

$3x_1 + 4x_2 + 6x_3 \ge 9$

$x_1 + 3x_2 + 2x_3 \ge 5$

$7x_1 - 2x_2 - x_3 \le 10$

$x_1 - 2x_2 + 4x_3 \ge 4$

$2x_1 + 5x_2 - 3x_3 \ge 3$

and $x_1, x_2, x_3 \ge 0.$

Solution

To make the dual in case of minimization, all constraints should be '≥' type. Hence we multiply the third constraint by –1 and we get,

Min $z = 5x_1 - 6x_2 + 4x_3$

Subject to the constraints

$3x_1 + 4x_2 + 6x_3 \ge 9$

$x_1 + 3x_2 + 2x_3 \ge 5$

$-7x_1 + 2x_2 + x_3 \ge -10$

$x_1 - 2x_2 + 4x_3 \ge 4$

$2x_1 + 5x_2 - 3x_3 \ge 3$

and $x_1, x_2, x_3 \ge 0.$

The primal problem in the matrix form is

$AX \ge b, b \in R^m, x \ge 0$

where,

$$A = \begin{pmatrix} 3 & 4 & 6 \\ 1 & 3 & 2 \\ -7 & 2 & 1 \\ 1 & -2 & 4 \\ 2 & 5 & -3 \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \; b = \begin{pmatrix} 9 \\ 5 \\ -10 \\ 4 \\ 3 \end{pmatrix}$$

The Dual of the above primal is obtained as follows:

$A^T Y \le C^T$, $C \in R^n$, $Y \ge 0,$

where,

$$A^T = \begin{pmatrix} 3 & 4 & -7 & 1 & 2 \\ 4 & 3 & 2 & -2 & 5 \\ 6 & 2 & 1 & 4 & -3 \end{pmatrix}, \; Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}, \; C^T = \begin{pmatrix} 5 \\ -6 \\ 4 \end{pmatrix}$$

Hence, the dual of an LPP is given by

Max $z = 9y_1 + 5y_2 - 10y_3 + 4y_4 + 3y_5$

Subject to the constraints

$3y_1 + 4y_2 - 7y_3 + y_4 + 2y_5 \le 5$

$4y_1 + 3y_2 + 2y_3 - 2y_4 + 5y_5 \le -6$

$6y_1 + 2y_2 + y_3 + 4y_4 - 3y_5 \le 4$

and $y_1, y_2, y_3, y_4, y_5 \ge 0.$

Example 3

Find the Dual of the following LPP:

Max $z = 12x_1 + 15x_2 + 9x_3$

Subject to the constraints

$8x_1 + 16x_2 + 12x_3 \le 25$

$4x_1 + 8x_2 + 10x_3 \ge 80$

$7x_1 + 9x_2 + 8x_3 = 105$

and $x_1, x_2, x_3 \ge 0.$

Solution

To write the dual in case of maximization problem, all the constraints should be of '≤' type. Hence, we multiply the second constraint by –1 and divide the third constraint into two parts i.e.,

$$\boxed{7x_1 + 9x_2 + 8x_3 = 105} \rightarrow \begin{cases} \boxed{7x_1 + 9x_2 + 8x_3 \le 105} \\ \boxed{7x_1 + 9x_2 + 8x_3 \ge 105} \end{cases}$$

The constraint $7x_1 + 9x_2 + 8x_3 \ge 105$ is violating the condition of being '$\le$', so we multiply this constraint by –1 on both sides. Hence the constraint is given as –

$-7x_1 - 9x_2 - 8x_3 \le -105.$

Hence, $7x_1 + 9x_2 + 8x_3 = 105$ is written as:

$7x_1 + 9x_2 + 8x_3 \le 105$ and $-7x_1 - 9x_2 - 8x_3 \le -105.$

Hence, the primal of the LPP is given by,

Max $z = 12x_1 + 15x_2 + 9x_3.$

Subject to the constraints:

$8x_1 + 16x_2 + 12x_3 \le 25$

$4x_1 + 8x_2 + 10x_3 \ge 80 \;\Rightarrow\; -4x_1 - 8x_2 - 10x_3 \le -80$

$7x_1 + 9x_2 + 8x_3 \le 105$

$7x_1 + 9x_2 + 8x_3 \ge 105 \Rightarrow -7x_1 - 9x_2 - 8x_3 \le -105$

and $x_1, x_2, x_3 \ge 0.$

The Dual of an LPP is

Min $z = 25y_1 - 80y_2 + 105y_3 - 105y_4$

Subject to the constraints:

$8y_1 - 4y_2 + 7y_3 - 7y_4 \ge 12$

$16y_1 - 8y_2 + 9y_3 - 9y_4 \ge 15$

$12y_1 - 10y_2 + 8y_3 - 8y_4 \ge 9$

and $y_1, y_2, y_3, y_4 \ge 0.$

In primal, we find three constraints and three variables. While in dual there are three constraints and four variables. So, in order to remove inconsistency, let $y_3 - y_4 = y_5$. Thus,

Min $z = 25y_1 - 80y_2 + 105y_5$

Subject to the constraints

$$8y_1 - 4y_2 + 7y_5 \geq 12$$

$$16y_1 - 16y_2 + 9y_5 \geq 15$$

$$12y_1 - 10y_2 + 8y_5 \geq 9$$

and $y_1, y_2 \geq 0$; y_5 is unrestricted in sign.

Example 4

Write the Dual of the following LPP:

Max $z = x_1 + x_2 + x_3$

Subject to the constraints

$$x_1 - 3x_2 + 4x_3 = 5$$

$$x_1 - 2x_2 \geq 3$$

$$2x_2 - x_3 \geq 4$$

and $x_1, x_2 \geq 0,$ x_3 is unrestricted in sign.

Solution

Since x_3 is unrestricted in sign, let $x_3 = x_4 - x_5$ such that $x_4 \geq 0, x_5 \geq 0$. So,

Min $z = x_1 + x_2 + x_4 - x_5$

Subject to the constraints

$$x_1 - 3x_2 + 4x_4 - 4x_5 = 5$$

$$x_1 - 2x_2 \geq 3$$

$$2x_2 - x_4 + x_5 \geq 4$$

$$x_1, x_2, x_4, x_5 \geq 0.$$

First constraint is written as:

$x_1 - 3x_2 + 4x_4 - 4x_5 \geq 5$ and

$x_1 - 3x_2 + 4x_4 - 4x_5 \leq 5$ or $-x_1 + 3x_2 - 4x_4 + 4x_5 \geq -5.$

Hence the primal is

Min $z = x_1 + x_2 + x_4 - x_5.$

Subject to the constraints:

$$x_1 - 3x_2 + 4x_4 - 4x_5 \geq 5$$

$$-x_1 + 3x_2 - 4x_4 + 4x_5 \geq -5$$

$$x_1 - 2x_2 \geq 3$$

$$2x_2 - x_4 + x_5 \geq 4$$

$$x_1, x_2, x_4, x_5 \geq 0.$$

The dual of the above primal is as follows:

Max $z = 5y_1 - 5y_2 + 3y_3 + 4y_4$

Subject to the constraints:

$$y_1 - y_2 + y_3 + 0y_4 \le 1$$

$$-3y_1 + 3y_2 - 2y_3 + 2y_4 \le 1$$

$$4y_1 - 4y_2 + 0y_3 - y_4 \le 1$$

$$-4y_1 + 4y_2 + 0y_3 + y_4 \le -1$$

and $y_1, y_2, y_3, y_4 \ge 0$

Hence,

Max $z = 5y_1 - 5y_2 + 3y_3 + 4y_4$

Subject to the constraints

$$y_1 - y_2 + y_3 + 0y_4 \le 1$$

$$-3y_1 + 3y_2 - 2y_3 + 2y_4 \le 1$$

$$4y_1 - 4y_2 + 0y_3 - y_4 = 1$$

and $y_1, y_2, y_3, y_4 \ge 0.$

Example 5

Write the dual of the following LPP and hence solve:

Min $z = 4x_1 + 6x_2$

Subject to the constraints

$$x_1 + 2x_2 \ge 80$$

$$3x_1 + x_2 \ge 75$$

and $x_1, x_2 \ge 0.$

Solution

The given LPP is in symmetric form.

Dual of the above problem is

Max $z = 80y_1 + 75y_2$

Subject to the constraints

$$y_1 + 3y_2 \le 4$$

$$2y_1 + y_2 \le 6$$

and $y_1, y_2 \ge 0.$

This dual can be solved using simplex method as follows:

After introducing the slack variables, the Dual LPP is

Max $z = 80y_1 + 75y_2 + 0S_1 + 0S_2$

Subject to the constraints

$$y_1 + 3y_2 + S_1 + 0S_2 = 4$$

$$2y_1 + y_2 + 0S_1 + S_2 = 6$$

and $y_1, y_2, S_1, S_2 \ge 0.$

The IBFS is obtained by setting the non-basic variables to zero i.e.,

$$y_1 = 0,\ y_2 = 0$$

Hence, the IBFS is

$$S_1 = 4, S_2 = 6.$$

Matrix form of constraint equations is

$$AX = B$$

where,

$$A = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix}, \; X = \begin{pmatrix} y_1 \\ y_2 \\ S_1 \\ S_2 \end{pmatrix}, \; B = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

With the IBFS, we move towards optimality using simplex table.

Simplex Table – 1

		$C_j \rightarrow$	80	75	0	0	
C_B	Basis	X_B	y_1	y_2	S_1	S_2	θ
0	S_1	4	**1**	3	1	1	4
0	S_2	6	**2***	1	0	0	**3** $\rightarrow$
	Z_j	0	0	0	0	0	
		$Z_j - C_j$	**–80** $\uparrow$	–75	0	0	

From the Simplex Table – 1, it is clear that, $Z_j - C_j$ is minimum for y_1 column. Hence, the variable y_1 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_2, indicating that S_2 should leave the basis.

Simplex Table – 2

		$C_j \rightarrow$	80	75	0	0	
C_B	Basis	X_B	y_1	y_2	S_1	S_2	θ
0	S_1	0	0	**5/2***	1	–1/2	**2/5** $\rightarrow$
80	y_1	3	1	1/2	0	1/2	6
	Z_j	240	80	40	0	40	
		$Z_j - C_j$	0	**–35** $\uparrow$	0	40	

From the Simplex Table – 2, it is clear that, $Z_j - C_j$ is minimum for y_2 column. Hence, the variable y_2 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_1, indicating that S_1 should leave the basis.

Simplex Table – 3

		$C_j \rightarrow$	80	75	0	0
C_B	Basis	X_B	y_1	y_2	S_1	S_2
75	y_2	2/5	0	1	2/5	–1/5
80	y_1	14/5	1	0	–1/5	3/5
	Z_j	254	80	75	14	33
		$Z_j - C_j$	0	0	14	33

Conclusion

From Simplex Table – 3, it is found that all $Z_j - C_j$ values are ≥ 0. Hence, the current solution is optimal. The values of primal variables x_1, x_2 correspond to slack variable of the dual problem. As seen from the above table, the net evaluation $Z_j - C_j$ corresponding to the variables S_1, S_2 are 14, and 33 respectively. Hence, the primal

Optimal Solution: $x_1 = 14$, $x_2 = 33$

Optimal Value: $Z = 254$.

Example 6

One unit of product A contributes Rs.7 and requires 3 units of raw material and 2 hours of labor. One unit of product B contributes Rs.5 and requires one unit of raw material and one hour of labor. Availability of raw material at present is 48 units and 40 hours of labor.

i. Formulate it as LPP.

ii. Write its dual.

iii. Solve the dual with simplex method and find the optimal product mix and shadow prices of the raw material and labor.

Solution

The mathematical formulation of LPP is

Max $z = 7x_1 + 5x_2$

Subject to the constraints

$3x_1 + x_2 \leq 48$

$2x_1 + x_2 \leq 40$

and $x_1, x_2 \geq 0$

where,

x_1, x_2 are the number of units of product A and B respectively.

Dual of the above symmetric LPP is

Min $z = 48y_1 + 40y_2$

Subject to the constraints

$3y_1 + 2y_2 \geq 7$

$y_1 + y_2 \geq 5$

and $y_1, y_2 \geq 0$

where,

y_1, y_2 are dual variables indicating the shadow prices of raw material and labor respectively.

Introducing Surplus Variables (S_1, S_2) with zero cost coefficients and Artificial Variables (A_1, A_2) with cost coefficients 'M', we get the standard form of LPP as –

Min $z = 48y_1 + 40y_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to the constraints

$$3y_1 + 2y_2 - S_1 + 0S_2 + A_1 + 0A_2 = 7$$

$$y_1 + y_2 + 0S_1 - S_2 + 0A_1 + A_2 = 5$$

and $y_1, y_2, S_1, S_2, A_1, A_2 \geq 0.$

The IBFS is obtained by considering the non-basic variables as zero i.e.,

$$y_1 = 0,\ y_2 = 0,\ S_1 = 0,\ S_2 = 0$$

Hence, the IBFS is

$$A_1 = 7, A_2 = 5$$

Matrix form of constraint equations is

$$AX = B$$

where,

$$A = \begin{pmatrix} 3 & 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 \end{pmatrix},\ X \begin{pmatrix} y_1 \\ y_2 \\ S_1 \\ S_2 \\ A_1 \\ A_2 \end{pmatrix},\ B = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

With the IBFS, we move towards optimality using simplex table.

Simplex Table – 4

		$C_j \rightarrow$	48	40	0	0	M	M	
C_B	Basis	X_B	y_1	y_2	S_1	S_2	A_1	A_2	θ
M	A_1	7	**3***	2	–1	0	1	0	**7/3 →**
M	A_2	5	1	1	0	–1	0	1	5
	Z_j	12M	4M	3M	–M	–M	M	M	
		$Z_j - C_j$	–48+4M↑	–40+3M	–M	–M	0	0	

From the Simplex Table – 4, it is clear that, highest $Z_j - C_j$ is for y_1 column. Hence, the variable y_1 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable A_1, indicating that A_1 should leave the basis.

Simplex Table – 5

		$C_j \rightarrow$	48	40	0	0	M	
C_B	Basis	X_B	y_1	y_2	S_1	S_2	A_2	θ
48	y_1	7/3	1	2/3*	–1/3	0	0	**7/2 →**
M	A_2	8/3	0	1/3	1/3	–1	1	5
	Z_j	$\frac{8M}{3}+112$	48	$\frac{M+96}{3}$	$\frac{M+48}{3}$	–M	M	
		$Z_j - C_j$	0	$\frac{-24+M}{3}$ ↑	$\frac{-48+M}{3}$	–M	0	

From the Simplex Table – 5, it is clear that, highest $Z_j - C_j$ is for y_2 column. Hence, the variable y_2 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable y_1, indicating that y_1 should leave the basis.

Simplex Table – 6

		$C_j \rightarrow$	48	40	0	0	M	
C_B	Basis	X_B	y_1	y_2	S_1	S_2	0	θ
40	y_2	7/2	3/2	1	–1/2	0	0	–7
M	A_2	3/2	–1/2	0	**1/2***	–1	1	**3 →**
	Z_j	$\frac{280+3M}{2}$	$\frac{120+3M}{2}$	40	$\frac{M+40}{2}$	–M	M	
		$Z_j - C_j$	$\frac{24-M}{2}$	0	$\frac{-40+M}{2}$ ↑	–M	0	

From the Simplex Table – 6, it is clear that, highest $Z_j - C_j$ is maximum for S_1 column. Hence, the variable S_1 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable A_2, indicating that A_2 should leave the basis.

Simplex Table – 7

		$C_j \rightarrow$	48	40	0	0
C_B	Basis	X_B	y_1	y_2	S_1	S_2
40	y_2	5	1	1	0	–1
0	S_1	3	–1	0	1	–2
	Z_j	200	40	40	0	–40
		$Z_j - C_j$	–8	0	0	–40

Conclusion

Solution of Dual

From the Simplex Table – 7, it is found that all $Z_j - C_j$ values are ≤ 0. Hence, the current solution is optimal. (since, the problem is of minimization). We obtain the optimum solution as y_1 = 0, y_2 = 5. Therefore, the shadow price of raw material is nil and that of labor is Rs.5 per hour.

Solution of Primal

From the Simplex Table – 4, it is found that all $Z_j - C_j$ values for S_1 and S_2. The optimal product mix is $x_1 = 0$, $x_2 = 40$ i.e., zero units of product A and 40 units of product B minimizes the cost to Rs.200.

Example 7

A television manufacturing company has two assembly plants, plant A and plant B, and two distribution outlets, outlet I and outlet II. Plant A can assemble 700 television sets a month and Plant B can assemble 900 Television sets per month. Outlet I must have atleast 500 televisions a month and outlet II must have atleast 1000 televisions a month. Transportation costs for shipping one television from each plant to each outlet are as follows. Rs.6 from plant A to outlet I, Rs.5 from plant A to outlet II, Rs.4 from plant B to outlet I, Rs.8 from plant B to outlet II. Find a shipping schedule that will minimize the total cost of shipping the televisions from the assembly plants to the distribution outlets. What is this minimum cost? Use the dual.

Solution

	Outlet I	Outlet II	Availability
Plant A	1 **6**	1 **5**	700
Plant B	1 **4**	1 **8**	900
Minimum requirement	500	1000	

Note: Numbers in bold letters indicate the cost, TV : Television.

Let x_1 be the number of TVs shipped from plant A to outlet I

Let x_2 be the number of TVs shipped from plant A to outlet II

Let x_3 be the number of TVs shipped from plant B to outlet I

Let x_4 be the number of TVs shipped from plant B to outlet II.

The mathematical formulation of the LPP is as follows:

$$\text{Min } Z = 6x_1 + 5x_2 + 4x_3 + 8x_4$$

Subject to

$$x_1 + x_2 \le 700$$

$$x_3 + x_4 \le 900$$

$$x_1 + x_3 \ge 500$$

$$x_2 + x_4 \ge 1000$$

$$x_1, x_2, x_3, x_4 \ge 0.$$

To solve the given non-symmetric LPP as dual firstly, we need to convert the primal into standard form.

Multiplying the constraints (i) and (ii) with '–' sign we get,

$$-x_1 - x_2 \ge -700$$

$$-x_3 - x_4 \ge -900$$

Hence the primal is

$$\text{Min } Z = 6x_1 + 5x_2 + 4x_3 + 8x_4$$

Subject to

$$-x_1 - x_2 \ge -700$$

$$-x_3 - x_4 \ge -900$$

$$x_1 + x_3 \ge 500$$

$$x_2 + x_4 \ge 1000$$

$$x_1, x_2, x_3, x_4 \ge 0.$$

The dual of the above Primal is

$$\text{Max } Z = -700y_1 - 900y_2 + 500y_3 + 1000y_4$$

Subject to

$$-y_1 + 0y_2 + y_3 + 0y_4 \le 6$$

$$-y_1 + 0y_2 + 0y_3 + y_4 \le 5$$

$$0y_1 - y_2 + y_3 + 0y_4 \le 4$$

$0y_1 - y_2 + 0y_3 + y_4 \le 8$

$y_1, y_2, y_3, y_4 \ge 0.$

Introducing slack variables, we have

Max $Z = -700y_1 - 900y_2 + 500y_3 + 1000y_4 + 0S_1 + 0S_2 + 0S_3 + 0S_4$

Subject to

$-y_1 + 0y_2 + y_3 + 0y_4 + S_1 + 0S_2 + 0S_3 + 0S_4 = 6$

$-y_1 + 0y_2 + 0y_3 + y_4 + 0S_1 + S_2 + 0S_3 + 0S_4 = 5$

$0y_1 - y_2 + y_3 + 0y_4 + 0S_1 + 0S_2 + S_3 + 0S_4 = 4$

$0y_1 - y_2 + 0y_3 + y_4 + 0S_1 + 0S_2 + 0S_3 + S_4 = 8$

$y_1, y_2, y_3, y_4, S_1, S_2, S_3, S_4 \ge 0.$

The IBFS is obtained by considering the non-basic variables as zero i.e.,

$y_1 = 0,\ y_2 = 0, y_3 = 0, y_4 = 0$

Hence, the IBFS is

$S_1 = 6, S_2 = 5,\ S_3 = 4,\ S_4 = 8$

Matrix form of constraint equations is

$AX = B$

where,

$$A = \begin{pmatrix} -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},\ X\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix},\ B = \begin{pmatrix} 6 \\ 5 \\ 4 \\ 8 \end{pmatrix}$$

With the IBFS, we move towards optimality using simplex table.

Simplex Table – 8

		$C_j \rightarrow$	–700	–900	500	1000	0	0	0	0	
C_B	Basis	X_B	y_1	y_2	y_3	y_4	S_1	S_2	S_3	S_4	θ
S_1	0	6	–1	0	1	0	1	0	0	0	∞
S_2	0	5	–1	0	0	**1***	0	1	0	0	**5** $\rightarrow$
S_3	0	4	0	–1	1	0	0	0	1	0	∞
S_4	0	8	0	–1	0	1	0	0	0	1	8
	Z_j		0	0	0	0	0	0	0	0	
		$Z_j - C_j$	700	900	–500	**–1000** $\uparrow$	0	0	0	0	

From the Simplex Table – 8, it is clear that, $Z_j - C_j$ is minimum for y_4 column. Hence, the variable y_4 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_2, indicating that S_2 should leave the basis.

Simplex Table – 9

		$C_j \rightarrow$	–700	–900	500	1000	0	0	0	0	
C_B	Basis	X_B	y_1	y_2	y_3	y_4	S_1	S_2	S_3	S_4	θ
S_1	0	6	–1	0	1	0	1	0	0	0	6
y_4	1000	5	–1	0	0	1	0	1	0	0	∞
S_3	0	4	0	–1	1*	0	0	0	1	0	4 $\rightarrow$
S_4	0	8	1	–1	0	1	0	–1	0	1	∞
	Z_j	5000	–1000	0	0	1000	0	1000	0	0	
		$Z_j - C_j$	–300	900	**–500** $\uparrow$	0	0	1000	0	0	

From the Simplex Table – 9, it is clear that, $Z_j - C_j$ is minimum for y_3 column. Hence, the variable y_3 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_3, indicating that S_3 should leave the basis.

Simplex Table – 10

		$C_j \rightarrow$	–700	–900	500	1000	0	0	0	0	
C_B	Basis	X_B	y_1	y_2	y_3	y_4	S_1	S_2	S_3	S_4	θ
S_1	0	2	–1	1	0	0	1	0	–1	0	–ve
y_4	1000	5	–1	0	0	1	0	1	0	0	–ve
y_3	500	4	0	–1	1	0	0	0	1	0	∞
S_4	0	3	**1***	–1	0	0	0	–1	0	1	**3** $\rightarrow$
	Z_j	7000	–1000	–500	500	1000	0	1000	500	0	
		$Z_j - C_j$	**–300** $\uparrow$	400	0	0	0	1000	500	0	

From the Simplex Table – 10, it is clear that, $Z_j - C_j$ is minimum for y_1 column. Hence, the variable y_1 should enter the basis.

Again from θ-rule, the ratio is minimum for the variable S_4, indicating that S_4 should leave the basis.

Simplex Table – 11

		$C_j \rightarrow$	–700	–900	500	1000	0	0	0	0
C_B	Basis	X_B	y_1	y_2	y_3	y_4	S_1	S_2	S_3	S_4
S_1	0	5	0	0	0	0	1	–1	–1	1
y_4	1000	8	0	–1	0	1	0	0	0	1
y_3	500	4	0	–1	1	0	0	0	1	0
y_1	–700	3	1	–1	0	0	0	–1	0	1
	Z_j	7900	–700	–800	500	1000	0	700	500	300
		$Z_j - C_j$	0	100	0	0	0	700	500	300

Conclusion

From the Simplex Table – 11, it is found that all $Z_j - C_j$ values are ≥ 0 . Hence, the current solution is optimal. The values of primal variables x_1, x_3, x_4 correspond to slack variable of the dual problem. As seen from the above table, the net evaluation $Z_j - C_j$ corresponding to the variables S_1, S_2, S_3 and S_4 are 0, 700, 500 and 300 respectively. Hence, the primal solution is as follows:

Optimal Solution: $x_1 = 0$, $x_2 = 700$, $x_3 = 500$, $x_4 = 300$

Optimal Value: $Z = 7900$.

SUMMARY

- If the primal variable corresponds to a slack/surplus variable in the dual problem, its optimum value is directly read from the net evaluation row $Z_j - C_j$ of the optimum simplex table of dual, as the net evaluations corresponding to the slack/surplus variable.
- If the primal variable corresponds to an artificial starting variable in the dual problem, its optimum value is directly read from the net evaluation row of the optimum simplex table of dual, as the net evaluations corresponding to this artificial variable, after deleting the constant M.
- The optimal solution to the primal (dual) gives directly an optimum solution to the dual (primal) problem.
- If the primal problem (or dual) has unbounded solution, then the corresponding dual (or primal) will have no feasible solution at all.
- Dual is easier to solve than primal.
- Duality theorem states that "Dual of a Dual is the Primal".
- Fundamental Duality theorem provides the basis foundation of duality in linear programming.
- If any variable of the primal problem is unrestricted in sign, the corresponding constraint in the dual will be a strict equality.
- If the *k*th constraint of the primal problem is equality, then the dual variable is unrestricted in sign.
- The existence theorem states that, if there does not exist any finite optimum solution to the primal (dual) problem, then there does not exist any feasible solution to the dual (primal) problem.
- The complementary slackness theorem states that, at optimality, if the primal constraint holds as a strict inequality, then the complementary dual constraint holds as equality. Conversely, if a dual constraint holds as a strict inequality, its complementary primal constraint holds as equality.

Exercise

1. What is duality in linear programming problem?
2. Show that dual of a dual is primal.
3. Explain the dual of a symmetric linear programming problem.
4. How do you read solution to the primal from the final simplex table of dual problem?
5. State and prove the complementary slackness theorem.
6. Explain the primal-dual relationship.

Write the dual to the following Linear Programming problems:

7. Maximize $Z = x_1 - x_2 + 3x_3$

   Subject to the constraints

   $x_1 + x_2 + x_3 \le 10$
   $2x_1 + 0x_2 - x_3 \le 2$
   $2x_1 - 2x_2 - 3x_3 \le 6$
   $x_1, x_2, x_3 \ge 0.$

8. Minimize $Z = 3x_1 - 2x_2 + 4x_3$

   Subject to the constraints

   $3x_1 + 5x_2 + 4x_3 \ge 7$
   $6x_1 + x_2 + 3x_3 \ge 4$
   $7x_1 - 2x_2 - x_3 \le 10$
   $x_1 - 2x_2 + 5x_3 \ge 3$
   $4x_1 + 7x_2 - 2x_3 \ge 2$
   $x_1, x_2, x_3 \ge 0.$

9. Minimize $Z = x_1 + 2x_2$

   Subject to the constraints

   $2x_1 + 4x_2 \le 160$
   $x_1 - x_2 = 30$
   $x_1 \ge 10$
   $x_1, x_2 \ge 0.$

10. Minimize $Z = x_1 - 3x_2 - 2x_3$

    Subject to the constraints

    $3x_1 - x_2 + 2x_3 \le 7$
    $2x_1 - 4x_2 \ge 12$
    $-4x_1 + 3x_2 + 8x_3 = 10$
    $x_1, x_2 \ge 0$, x_3 is unrestricted in sign.

11. Maximize $Z = x_1 - 2x_2 + 3x_3$

    Subject to the constraints

    $-2x_1 + x_2 + 3x_3 = 2$
    $2x_1 + 3x_2 + 4x_3 = 1$
    $x_1, x_2, x_3 \ge 0.$

Use duality to obtain an optimum solution to the following Linear Programming problems:

12. Maximize $Z = 2x_1 + 3x_2$

    Subject to the constraints

    $-x_1 + 2x_2 \le 4$
    $x_1 + x_2 \le 6$
    $x_1 + 3x_2 \le 9$
    $x_1, x_2 \ge 0.$

13. Minimize $z = 15x_1 + 10x_2$
Subject to the constraints
$3x_1 + 5x_2 \geq 5$
$5x_1 + 2x_2 \geq 3$
$x_1, x_2 \geq 0.$

14. Maximize $Z = 3x_1 + 2x_2$
Subject to the constraints
$x_1 + x_2 \geq 1$
$x_1 + x_2 \leq 7$
$x_1 + 2x_2 \leq 10$
$x_2 \leq 3$
$x_1, x_2 \geq 0.$

15. Maximize $z = 6x_1 + 4x_2 + 6x_3 + x_4$
Subject to the constraints
$4x_1 + 5x_2 + 4x_3 + 8x_4 = 21$
$3x_1 + 7x_2 + 8x_3 + 2x_4 \leq 48$
$x_1, x_2, x_3, x_4 \geq 0.$

16. Maximize $z = 6x + 5y - 3z - 4w$
Subject to the constraints
$2x + 3y + 2z - 4w = 24$
$x + 2y \leq 10$
$x + y + 2z + 3w \leq 15$
$y + z + w \leq 8$
$x, y, z, w \geq 0.$

17. Maximize $z = 4x_1 + 3x_2$
Subject to the constraints
$x_1 \leq 6$
$x_2 \leq 8$
$x_1 + x_2 \leq 7$
$3x_1 + x_2 \leq 15$
$-x_2 \leq 1$
$x_1, x_2 \geq 0.$

18. Maximize $z = x_1 + 5x_2$
Subject to the constraints
$3x_1 + 4x_2 \leq 6$
$x_1 + 3x_2 \leq 2$
$x_1, x_2 \geq 0.$

19. Minimize $z = 10y_1 + 6y_2 + 2y_3$
Subject to the constraints
$-y_1 + y_2 + y_3 \geq 1$
$3y_1 + y_2 - y_3 \geq 2$
$y_1, y_2, y_3 \geq 0.$

<u>**Chapter VI**</u>

Transportation Problem

After reading this chapter, you will be conversant with:

- Methods for Obtaining IBFS
  - North-West Corner Method
  - Least Cost Entry Method
  - Vogel's Approximation Method
- Test for Optimality
  - Stepping Stone Method
  - MODI Method
- Profit Maximization in Transportation Problem
- Time-Minimizing Transportation Problems
- Transshipment Problem

Introduction

The Transportation Problem (TP) is one of the types of LPP, in which the objective is to transport various quantities of a single homogeneous commodity to different destinations in such a way that the total transportation cost is minimum. Transportation problems give direct relevance to decisions in the area of distribution policy making, where the objective is minimization of transportation cost. The various features of LPP can be observed in these problems. Here, the availability as well as requirements of the various centers are finite and constitute the limited resources. It is also assumed that cost of shipping is linear. Thus, these problems could also be solved by "Simplex Method".

The Transportation Problem was first formulated by F L Hitchcock in 1941, and was discussed in detail by the Nobel Laureate T C Koopmans in 1947. The LPP formulation of the transportation problem and the solution procedure was first given by Dantzig in 1951.

In this Chapter, the methods of finding initial and optimum solution to TP's are discussed with illustrations.

Assumptions

The following are some basic assumptions of the transportation model:

i. **Availability of the Quantity:** Quantity available for the distribution at different sources or depots is equal to total requirement of different consumption centers i.e., quantity available = quantity required.

ii. **Transportation of the Items:** Items can be conveniently transported from every production center to every consumption center.

iii. **Cost per Unit:** The per unit transportation cost of items from one production center to another consumption center is certain.

iv. **Independent Cost:** The per unit cost of transportation is independent of the quantity dispatched.

v. **Objective:** The objective of such an arrangement is to minimize the total cost of transportation for the organization as a whole.

Definitions

The following terms may be defined with reference to the Transportation Problem:

Feasible Solution (FS): A set of non-negative individual allocations $(x_{ij} \geq 0)$ which simultaneously removes deficiencies is called a feasible solution.

Basic Feasible Solution (BFS): A feasible solution to a m-origin, n-destination problem is said to be basic if the number of positive allocations are $m + n - 1$ i.e., one less than the sum of rows and columns.

If the number of allocations in a BFS is less than $m + n - 1$, it is called degenerate BFS (otherwise, non-degenerate BFS).

Optimal Solution: A feasible solution is said to be optimal, if it minimizes the total transportation cost. The optimal solution itself may or may not be a basic solution. This is done through successive improvements to the IBFS until no further decrease in transportation cost is possible.

Tabular Representation of the Model

Suppose there are m factories and n warehouses. The transportation problem is usually represented in a tabular form as given in the following table:

Table I (Unit Transportation Costs)

Factory	Warehouse						Factory Capacities
	W_1	W_2	$\cdots$	W_j	$\cdots$	W_n	
F_1	C_{11}	C_{12}	$\cdots$	C_{1j}	$\cdots$	C_{1n}	a_1
F_2	C_{21}	C_{22}	$\cdots$	C_{2j}	$\cdots$	C_{2n}	a_2
F_3	C_{31}	C_{32}	$\cdots$	C_{3j}	$\cdots$	C_{3n}	a_3
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
F_i	C_{i1}	C_{i2}	$\cdots$	$\boldsymbol{C_{ij}}$	$\cdots$	C_{in}	a_i
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
F_m	C_{m1}	C_{m2}		C_{mj}		C_{mn}	a_m
Warehouse Capacities	b_1	b_2	$\cdots$	b_j		b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Table II (Number of Units Transported)

Factory	Warehouse						Factory Capacities
	W_1	W_2	$\cdots$	W_j	$\cdots$	W_n	
F_1	x_{11}	x_{12}	$\cdots$	x_{1j}	$\cdots$	x_{1n}	a_1
F_2	x_{21}	x_{22}	$\cdots$	x_{2j}	$\cdots$	x_{2n}	a_2
F_3	x_{31}	x_{32}	$\cdots$	x_{3j}	$\cdots$	x_{3n}	a_3
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
F_i	x_{i1}	x_{i2}	$\cdots$	$\boldsymbol{x_{ij}}$	$\cdots$	x_{in}	a_i
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
F_m	x_{m1}	x_{m2}		x_{mj}		x_{mn}	a_m
Warehouse Capacities	b_1	b_2	$\cdots$	b_j		b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Tables I and II put together can be represented as follows:

Table III

To / From	W_1	W_2	$\cdots$	W_n	Supply a_i
F_1	c_{11} $\boxed{x_{11}}$	c_{12} $\boxed{x_{12}}$	$\cdots$	c_{1n} $\boxed{x_{1n}}$	a_1
F_2	c_{21} $\boxed{x_{21}}$	c_{22} $\boxed{x_{22}}$	$\cdots$	c_{2n} $\boxed{x_{2n}}$	a_2
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
F_m	c_{m1} $\boxed{x_{m1}}$	c_{m2} $\boxed{x_{m2}}$	$\cdots$	c_{mn} $\boxed{x_{mn}}$	a_m
Demand b_j	b_1	b_2	$\cdots$	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

C_{ij} : Cost incurred in transporting one unit from *i*th factory (origin) to *j*th warehouse (destination).

x_{ij} : Number of units transported from *i*th factory (origin) to *j*th warehouse (destination).

Table III indicates that there are ($m + n$) constraints and $m \times n$ variables. Since all ($m + n$) constraints are equations, there is one extra (redundant) equation on account of two way rim (supply and demand) conditions. The extra constraint can be derived from the other constraints without affecting the feasible solution. It follows that, any feasible solution for a transportation problem must have exactly ($m + n - 1$) non-negative basic decision variables x_{ij} (or allocations) satisfying rim conditions.

Balanced and Unbalanced Transportation Problems

If $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_j$, then the problem is balanced, otherwise unbalanced. In an unbalanced transportation problem, if $\sum_{i=1}^{m} a_i < \sum_{i=1}^{n} b_j$, then introduce a dummy row in which all costs are assumed to be zeros, and its corresponding availability (capacity) is considered as $\sum_{i=1}^{n} b_j - \sum_{i=1}^{m} a_i$.

Similarly if $\sum_{i=1}^{n} b_j < \sum_{i=1}^{m} a_i$, then introduce a dummy column with zero costs and its corresponding requirement as $\sum_{i=1}^{m} a_i - \sum_{i=1}^{n} b_j$.

Remarks:

i. When the number of positive allocations (values of decision variables) at any stage of the feasible solution is less than the required number (rows + columns – 1) i.e., number of independent constraint equations, the solution is said to be degenerate, otherwise non-degenerate.

ii. Cells in the transportation table having positive allocation will be called occupied cells, otherwise empty or non-occupied cells.

Mathematical Formulation

Let $S_1, S_2, ..., S_m$: denote '*m*' as origins (sources).

$D_1, D_2, ..., D_n$: denote '*n*' destinations.

a_i : denote number of units available at *ith* origin (i = 1, 2, 3… m)

b_j : denote number of units required at *j*th destination (j = 1, 2, 3… n)

C_{ij} : Cost incurred in transporting one unit from *i*th factory (origin) to *j*th warehouse (destination).

x_{ij} : number of units transported from *i*th factory (origin) to *j*th warehouse (destination). ($x_{ij} \geq 0$)

The problem is to determine non-negative values of x_{ij} satisfying both the availability constraint,

$$\sum_{j=1}^{m} x_{ij} = a_i \quad \text{for } i=1,2,...,m.$$

as well as requirement constraint: $\sum_{i=1}^{m} x_{ij} = b_j$ for j = 1,2.....n .

It is also assumed that total availability $\sum a_i$ satisfy the total requirement $\sum b_j$, i.e., $\sum a_i = \sum b_j$ … (1)

$$\left[\begin{array}{l}\text{In case } \sum a_i \neq \sum b_j \text{ some manipulation is required,} \\ \text{to make} \sum a_i = \sum b_j \text{, this will be discussed later}\end{array}\right].$$

TP in the Form of LPP

The LPP is to determine non-negative (≥ 0) values of x_{ij} satisfying the availability constraints:

$$\sum_{j=1}^{m} x_{ij} = a_i \quad \text{for } i = 1,2,...,m \qquad \text{… (2)}$$

as well as requirement constraint:

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \text{for } j = 1,2,...,n \qquad \text{… (3)}$$

such that the total cost of shipping

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij} \qquad \text{… (4)}$$

is minimum.

It may be observed that the constraint equations (2), (3) and the objective function (4) are all linear in x_{ij}. So, it may be viewed as a linear programming problem.

or

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

Subject to the constraints

$$\sum_{j=1}^{m} x_{ij} = a_i \quad \text{for } i = 1,2,...,m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \text{for } j = 1,2,...,n$$

$$x_{ij} \geq 0.$$

Particular Case: The assignment problem is seen to be a special case of transportation problem when each origin is associated with one and only one destination. In such case, $m = n$ and the numerical evaluations of such associations are called ‘effectiveness’ instead of ‘transportation costs’. Mathematically, all a_i and b_j are unity and each x_{ij} is limited to one of the two values, 0 and 1. In such circumstances, exactly n of the x_{ij} can be non-zero (i.e., unity), one for each origin and one for each destination.

Example 1

A company has three factories S_1, S_2, S_3 with production capacity of 7, 9 and 18 units (in 100’s) per week of a product, respectively. These units are to be shipped to four warehouses D_1, D_2, D_3 and D_4 with requirement 5, 8, 7 and 14 units (in 100's) per week, respectively. The transportation costs (in rupees) per unit between factories to warehouses are given in the table below:

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

Formulate this transportation problem as an LPP to minimize the total transportation cost.

Solution

Formulation

Let x_{ij} = number of units of the product to be transported from factory i (i =1, 2, 3) to warehouse j (j =1, 2, 3, 4).

The transportation problem is stated as an LP model as follows:

Minimize (total transportation cost)

$$Z = 19x_{11} + 30x_{12} + 50x_{13} + 10x_{14} + 70x_{21} + 30x_{22}$$
$$+ 40x_{23} + 60x_{24} + 40x_{31} + 8x_{32} + 70x_{33} + 20x_{34}$$

Subject to the constraints

i. Capacity constraints

$$x_{11} + x_{12} + x_{13} + x_{14} = 7$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 9$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 18.$$

ii. Requirement constraints

$$x_{11} + x_{21} + x_{31} = 5$$
$$x_{12} + x_{22} + x_{32} = 8$$
$$x_{13} + x_{23} + x_{33} = 7$$
$$x_{14} + x_{24} + x_{34} = 14$$

and $x_{ij} \geq 0$ for all i and j.

In the above LP model, there are $m \times n = 3 \times 4 = 12$ decision variables, x_{ij} and $m + n = 7$ constraints, where, m is the number of rows and n is the number of columns in a general transportation table.

Transportation Algorithm

The solution to a transportation algorithm is as given below:

Step 1: *Formulate the problem and set up in the matrix form*

The formulation of the transportation problem is similar to the LP formulation. Here, the objective function is the total transportation cost and the constraints are the supply and demand available at each source and destination, respectively.

Step 2: *Check whether the transportation problem is balanced or not*

If it is a balanced TP, go to step 3. Otherwise, convert it into a balanced TP by adding a dummy supply centre (row) or a dummy demand center (column) as the need arises, and proceed to step 3.

Step 3: *Obtain an Initial Basic Feasible Solution (IBFS)*

There are three methods to obtain an initial solution.

i. North-West Corner Method (NWCM)

ii. Least Cost Entry Method (LCEM)

iii. Vogel's Approximation Method (VAM).

The initial basic feasible solution obtained by any one of the three methods must satisfy the following conditions:

i. The solution must be feasible, i.e., it must satisfy all the supply and demand constraints.

ii. The number of positive allocations must be equal to $m + n - 1$, where, m is the number of rows and n is the number of columns.

Any solution that satisfies the above conditions is called non-degenerate basic feasible solution, otherwise degenerate solution.

Step 4: *Test the solution for optimality*

Here, we discuss two methods to reach the optimality.

i. Stepping Stone Method (SSM)

ii. MODI (Modified Distribution Method).

If the solution is optimal then stop. Otherwise, determine a new improved solution.

Step 5: *Update the solution*

Repeat Step 4 until an optimal solution is reached.

METHODS FOR OBTAINING IBFS

In this Chapter, we study three methods for obtaining the initial basic feasible solution:

i. North-West Corner Method (NWCM)

ii. Least Cost Entry Method (LCEM)

iii. Vogel's Approximation Method (VAM).

North-West Corner Method (NWCM)

This is the most systematic and easiest method for obtaining the initial basic feasible solution. Steps involved in this method are:

Step 1: Construct an empty $m \times n$ matrix, with rows and columns.

Step 2: Indicate the row totals and column totals at the end of the corresponding rows and columns.

Step 3: Starting with (1,1) cell at the North-West Corner of the matrix, allocate maximum possible quantity keeping in view that allocation can neither be more than the quantity required by the respective warehouses nor more than the quantity available at each supply center.

Step 4: Adjust the supply and demand numbers in the respective row and column allocations.

Step 5: i. If the supply for the first row is exhausted, then move down to the first cell in the second row and first column and go to step 4.

ii. If the demand for the first column is satisfied, then move to the next cell in the second column and first row and go to step 4.

iii. If for any cell supply equals to demand, then the next allocation can be made in cell either in the next row or next column.

Step 6: Continue the procedure until the total available quantity is fully allocated to the cells required.

Example 2

Find the IBFS for the following TP using NWCM.

Plant \ Warehouse	W_1	W_2	W_3	Supply S_i
P_1	7	6	9	**20**
P_2	5	7	3	**28**
P_3	4	5	8	**17**
Demand D_j	**21**	**25**	**19**	**65**

Solution

The given transportation problem is a balanced transportation problem, since, $\sum S_i = \sum D_j$ =65. Hence, we find IBFS using NWCM.

We start with first north-west cell (P_1, W_1) and allocate the min (S_1, D_1) = min (20, 21) = 20. Therefore, we allocate 20 units to this cell which completely exhausts the supply at plant P_1 and leaves a balance of (21 – 20) = 1 unit of demand at warehouse W_1.

	W_1	W_2	W_3	S_i
P_1	7 $\boxed{20}$	6	9	**20**
P_2	5	7	3	**28**
P_3	4	5	8	**17**
D_j	**~~21~~=1**	**25**	**19**	

Now, we move vertically downward to the next north-west cell (P_2, W_1). At this stage, the largest allocation possible is the min $(S_2, D_1 - 20)$ = min (28, 1) = 1. This allocation of 1 unit to the cell (P_2, W_1) completely satisfies the demand of the warehouse W_1. However, this leaves a balance of (28 – 1) = 27 units of supply at plant P_2.

	W_1	W_2	W_3	S_i
P_2	5 $\boxed{1}$	7	3	**~~28~~ =27**
P_3	4	5	8	**17**
D_j	–	**25**	**19**	

Now, in the remaining table we move again horizontally to the north-west cell (P_2, W_2). Since the demand of warehouse W_2 is 25 units while supply available at plant P_2 is 27 units, therefore, the min (27, 25) = 25 units are allocated to the cell (P_2, W_2). The demand of warehouse W_2 is now satisfied and a balance of (27 – 25) = 2 units of supply remain at plant P_2.

	W_2	W_3	S_i
P_2	7 $\boxed{25}$	3	~~**27**~~ **=2**
P_3	5	8	**17**
D_j	–	**19**	

Moving again horizontally, we allocate two units to the north-west cell (in the remaining table) (P_2, W_3).

Since the demand of warehouse W_3 is 19 units while supply available at plant P_2 is 2 units, therefore, the min (19, 2) = 2 units are allocated to the cell (P_2, W_3).

At this stage, 17 units are available at plant P_3 and 17 units are required at warehouse W_3.

	W_3	S_i
P_2	3 $\boxed{2}$	–
P_3	8	**17**
D_j	~~**19**~~ **=17**	

Now we move vertically downward to the last possible north-west cell (P_3, W_3). At this cell, 17 units are available at plant P_3 and 17 units are required at warehouse W_3. So, we allocate 17 units to this cell (P_3, W_3).

	W_3	S_i
P_3	8 $\boxed{17}$	–
D_j	–	

Hence, we have made all the allocations. It may be noted here that, there are 5 = (3 + 3 – 1) allocations which are necessary to proceed further.

Conclusion

The initial basic feasible solution using NWCM is shown below in the table:

	W_1	W_2	W_3	S_i
P_1	7 $\boxed{20}$	6	9	**20**
P_2	5 $\boxed{1}$	7 $\boxed{25}$	3 $\boxed{2}$	**28**
P_3	4	5	8 $\boxed{17}$	**17**
D_j	**21**	**25**	**19**	**65**

The numbers of allocated cells are five which is equal to the required number $(m + n - 1) = (3 + 3 - 1 = 5)$. Thus, this solution is a non-degenerate solution.

The total cost is obtained by multiplying each x_{ij} in occupied cells with the corresponding c_{ij} and adding as follows:

The total transportation cost for this initial solution

$= 20\times7+1\times5+25\times7+2\times3+17\times8 = \text{Rs.}462.$

Example 3

Find IBFS to solve the following TP using NWCM.

Plant \ Market	A	B	C	Production at Plant
X	11	21	16	**14**
Y	07	17	13	**26**
Z	11	23	21	**36**
Market Requirement	**18**	**28**	**25**	

Solution

The given transportation problem is an unbalanced transportation problem, since, $\sum a_i \neq \sum b_j$ where, $\sum a_i = 76; \sum b_j = 71$. So, it should be balanced by introducing Dummy column with **zero** cost.

Plant \ Market	A	B	C	Dummy	Production at Plant
X	11	21	16	0	**14**
Y	07	17	13	0	**26**
Z	11	23	21	0	**36**
Market Requirement	**18**	**28**	**25**	**5**	**71**

Now, we find IBFS using NWCM.

We start with the North-West Corner cell (X, A). Here, we find production of 14 units at plant X whereas, the requirement is 18 units. Hence, we allocate min (14, 18) = 14 units at the cell (X, A).

Plant \ Market	A	B	C	Dummy	Production at Plant
X	11 $\boxed{14}$	21	16	0	~~**14**~~
Y	07	17	13	0	**26**
Z	11	23	21	0	**36**
Market Requirement	~~**18**~~ =4	**28**	**25**	**5**	

Next we move towards the North-West Corner cell (Y, A). Here, the production at plant Y is 26 units, whereas, the requirement is 4 units. Hence, we allocate min (26, 4) = 4 units to the cell (Y, A).

Plant \ Market	A	B	C	Dummy	Production at Plant
Y	7 $\boxed{4}$	17	13	0	~~**26**~~ = 22
Z	11	23	21	0	**36**
Market Requirement	--	**28**	**25**	**5**	

Next we move towards the North-West Corner cell (Y, B). Here, the production at plant Y is 22 units, whereas, the requirement is 28 units. Hence, we allocate min (22,28) = 22 units to the cell (Y, B).

Plant \ Market	B	C	Dummy	Production at Plant
Y	17 $\boxed{22}$	13	0	--
Z	23	21	0	**36**
Market Requirement	**~~28~~** = 6	**25**	**5**	

Next we move towards the North-West Corner cell (Z, B). Here, the production at plant Z is 30 units, whereas, the requirement is 6 units. Hence, we allocate min (30, 6) = 6 units to the cell (Z, B).

Plant \ Market	B	C	Dummy	Production at Plant
Z	23 $\boxed{6}$	21	0	**~~36~~** = 30
Market Requirement	--	**25**	**5**	

Next we move towards the North-West Corner cell (Z, C). Here, the production at plant Z is 30 units, whereas, the requirement is 25 units. Hence, we allocate min (30, 25) = 25 units to the cell (Z, C).

Plant \ Market	C	Dummy	Production at Plant
Z	21 $\boxed{25}$	0	**~~30~~** = 5
Market Requirement	--	**5**	

Next we move towards the North-West Corner cell (Z, Dummy). Here, the production at plant Z is 5 units, and also the requirement is 5 units. Hence, we allocate 5 units to the cell (Z, Dummy).

Plant \ Market	Dummy	Production at Plant
Z	0 $\boxed{5}$	–
Market Requirement	–	

Conclusion

The number of allocated cells are six, which is equal to the required number ($m + n - 1$) = (4 + 3 – 1 = 6). Thus, this solution is a non-degenerate solution.

Hence, the initial feasible solution is obtained as below:

Plant \ Market	A	B	C	Dummy	Production at Plant
X	11 $\boxed{14}$	21	16	0	**14**
Y	7 $\boxed{4}$	17 $\boxed{22}$	13	0	**26**
Z	11	23 $\boxed{6}$	21 $\boxed{25}$	0 $\boxed{5}$	**36**
Market Requirement	**18**	**28**	**25**	**5**	

Total transportation cost is

$$11 \times 14 + 7 \times 4 + 17 \times 22 + 23 \times 6 + 21 \times 25 + 0 \times 5 = \text{Rs.}1,219.$$

Least Cost Entry Method (LCEM)

This method is also known as Matrix Minima Method (MMM). This method takes into consideration the least cost and therefore, takes less time to solve the problem. The procedure is given below:

Step 1: Select the cell with the least transportation cost among all the rows or columns of the transportation table. If the minimum cost is not unique, then select arbitrarily any cell with the least cost.

Step 2: Allocate as many units as possible, to the cell determined in step 1 and eliminate that row/column in which either capacity or requirement is exhausted.

Step 3: Adjust the capacity and requirement for the next allocations.

Step 4: Repeat steps 1 to 3 for the reduced table until the entire capacities are exhausted to fill the requirement at different destinations.

Example 4

Find the IBFS to solve the following TP using LCEM.

From \ To	D_1	D_2	D_3	D_4	Supply
S_1	19	20	50	10	**7**
S_2	70	30	40	60	**9**
S_3	40	8	70	20	**18**
Demand	**5**	**8**	**7**	**14**	**34**

Solution

The given transportation problem is a balanced transportation problem, since,

$\sum a_i = \sum b_j$ = 34. Hence, we find IBFS using LCEM.

The cell with least cost (i.e., 8) is (S_3, D_2). The maximum which we can allocate to this cell is min (8,18) = 8 units. This meets the complete demand at D_2 and leaves 10 units with S_3 as shown in the table below:

From \ To	D_1	D_2	D_3	D_4	Supply
S_1	19	20	50	10	**7**
S_2	70	30	40	60	**9**
S_3	40	8 $\boxed{8}$	70	20	~~**18**~~ **=10**
Demand	**5**	–	**7**	**14**	**34**

In the reduced table without column D_2, the next smallest unit transportation cost is 10 in cell (S_1, D_4). The maximum which can be allocated to this cell is 7. This exhausts the capacity at S_1 and leaves 7 units with D_4 as unsatisfied demand as shown in the table below:

To / From	D_1	D_3	D_4	Supply
S_1	19	50	10 $\boxed{7}$	–
S_2	70	40	60	**9**
S_3	40	70	20	**10**
Demand	**5**	**7**	~~**14**~~ **=7**	**34**

The next smallest cost is 20 in cell (S_3, D_4). The maximum allocation in this cell could be 7 units. This satisfies the entire demand of D_4 and leaves 3 units with S_3 as remaining supply as shown in the table below:

To / From	D_1	D_3	D_4	Supply
S_2	70	40	60	**9**
S_3	40	70	20 $\boxed{7}$	~~**10**~~ **= 3**
Demand	**5**	**7**	–	**34**

The next least cost is not unique. That is, there are two cells (S_2, D_3) and (S_3, D_1) having the same unit transportation cost 40. Allocate 7 units in cell (S_2, D_3) because it can accommodate more units as compared to cell (S_3, D_1).

To / From	D_1	D_3	Supply
S_2	70	40 $\boxed{7}$	~~**9**~~ **= 2**
S_3	40	70	**3**
Demand	**5**	–	

Allocate 3 units (only supply left with S_3) in cell (S_3, D_1). The remaining demand of 2 units of D_1 is fulfilled from S_2.

To / From	D_1	Supply
S_2	70 $\boxed{2}$	–
S_3	40 $\boxed{3}$	–
Demand	–	

Since, supply and demand at each origin and destination are exhausted, the initial solution is arrived at as shown in the table.

Conclusion

The number of allocated cells are six which is equal to the required number $(m + n - 1) = (3 + 4 - 1 = 6)$. Thus, this solution is a non-degenerate solution.

To / From	D_1	D_2	D_3	D_4	Supply
S_1	19	20	50	10 $\boxed{7}$	**7**
S_2	70 $\boxed{2}$	30	40 $\boxed{7}$	60	**9**
S_3	40 $\boxed{3}$	8 $\boxed{8}$	70	20 $\boxed{7}$	**18**
Demand	**5**	**8**	**7**	**14**	**34**

The total transportation cost of the initial solution by LCEM is calculated as given below:

$$\text{Total cost} = 7\times10 + 2\times70 + 7\times40 + 3\times40 + 8\times8 + 7\times20 = \text{Rs.}814\,.$$

Example 5

Find the IBFS to solve the following TP using MMM.

Dealer / Plant	A	B	C	Availability
X	9	19	14	**12**
Y	5	15	11	**24**
Z	9	21	19	**32**
Requirement	**16**	**24**	**23**	

Solution

The given transportation problem is an unbalanced transportation problem, since, $\sum a_i \neq \sum b_j$ where, $\sum a_i = 68$; $\sum b_j = 63$. So, it should be balanced by introducing Dummy column with zero cost.

Dealer / Plant	A	B	C	Dummy	Availability
X	9	19	14	0	**12**
Y	5	15	11	0	**24**
Z	9	21	19	0	**32**
Requirement	**16**	**24**	**23**	**5**	

The cell with least cost (i.e., 0) is not unique. There are three unit cells with the same unit transportation cost 0 i.e., (X, Dummy), (Y, Dummy) and (Z, Dummy). Allocate 5 units in cell (Z, Dummy) because the availability at *Z* is more units as compared to *X* and *Y*.

The maximum allocation in the cell (Z, Dummy) could be 5 units. This satisfies the entire demand of Dummy and leaves 27 units with Z as remaining supply, as shown in the table below:

Dealer / Plant	A	B	C	Dummy	Availability
X	9	19	14	0	**12**
Y	5	15	11	0	**24**
Z	9	21	19	0 $\boxed{5}$	~~**32**~~ **= 27**
Requirement	**16**	**24**	**23**	–	

In the reduced table without column Dummy, the next least cost is 5 in the cell (Y, A). The maximum allocation in this cell could be 16 units. This satisfies the entire demand at A and leaves 8 units with Y as remaining supply as shown in the table below:

Dealer / Plant	A	B	C	Availability
X	9	19	14	**12**
Y	5 $\boxed{16}$	15	11	~~**24**~~ **=8**
Z	9	21	19	**27**
Requirement	–	**24**	**23**	

The next least cost is 11 in cell (Y, C). The maximum allocation in this cell could be 8 units. This satisfies the entire availability at Y and leaves 15 units with Y as remaining supply as shown in the table below:

Dealer / Plant	B	C	Availability
X	19	14	**12**
Y	15	11 $\boxed{8}$	–
Z	21	19	**27**
Requirement	**24**	~~**23**~~ **=15**	

The next least cost is 14 in cell (X, C). The maximum allocation in this cell could be 12 units. This satisfies the entire availability at X and leaves 3 units with C as remaining supply as shown in the table below:

Dealer / Plant	B	C	Availability
X	19	14 $\boxed{12}$	–
Z	21	19	**27**
Requirement	**24**	~~**15**~~ **=3**	

The next least cost is 19 in cell (Z, C). The maximum allocation in this cell could be 3 units. This satisfies the entire demand at C and leaves 24 units with Z as remaining supply and also remaining 24 units to the cell (Z, B) as shown in the table below:

Dealer / Plant	B	C	Availability
Z	21 $\boxed{24}$	19 $\boxed{3}$	–
Requirement	–	–	

Since, supply and demand at each origin and destination are exhausted, the initial solution is arrived at as shown in the following table:

Dealer / Plant	A	B	C	Dummy	Availability
X	9	19	14 $\boxed{12}$	0	**12**
Y	5 $\boxed{16}$	15	11 $\boxed{8}$	0	**24**
Z	9	21 $\boxed{24}$	19 $\boxed{3}$	0 $\boxed{5}$	**32**
Requirement	**16**	**24**	**23**	**5**	

The number of allocated cells are six which is equal to the required number $(m + n - 1) = (3 + 4 - 1 = 6)$. Thus, this solution is a non-degenerate solution.

The total transportation cost of the initial solution by LCEM is calculated as given below:

$$\text{Total cost} = 14\times12+5\times16+11\times8+21\times24+19\times3+0\times5 = \text{Rs.}897 .$$

Vogel's Approximation Method (VAM)

This method is preferred over the other two methods because the initial basic feasible solution obtained with VAM is either optimal or very close to the optimal solution. Therefore, the amount of time required to calculate the optimum solution is reduced. In Vogel's approximation method, the basis for allocations is unit cost penalty. Penalty is the difference between the least and the next least cost. We select the row/column with highest penalty for allocation. The subsequent allocation in cells are done keeping in view the least unit cost penalty. Various steps in the iteration process are as follows:

Step 1: Construct the matrix with unit transportation costs, availabilities and requirements.

Step 2: Compute a penalty for each row and column in the transportation table. The penalty for a given row and column is merely the difference between the smallest cost and the next smallest cost element in that particular row or column.

Step 3: Identify the row/column with largest penalty. In the identified row (column), choose the cell which has the smallest cost and allocate the maximum possible quantity to this cell. Delete the row (column) in which availability (requirement) is exhausted.

Whenever the largest penalty among rows and columns is not unique, make an arbitrary choice.

Step 4: Repeat step 1 to 3 for the reduced table until the entire availabilities are used to fill the requirement at different destination requirements.

Step 5: From step 4 we will get Initial Feasible Solution (IFS). Now, for IBFS find the total transportation cost by multiplying the cell allocations by corresponding unit cost.

Though this method takes more time as compared to other methods, but still it gives better solutions, and saves more time in reaching the optimal solution.

Example 6

Find the IBFS to solve the following TP using VAM.

The manufacturer wants to ship 8 loads of his product from production centers X, Y and Z to distribution centers A, B and C. The mileage from various origins to various destinations is given in the following matrix.

Production \ Distribution	A	B	C	Availability
X	50	30	220	**1**
Y	90	45	170	**3**
Z	250	200	50	**4**
Requirement	**3**	**3**	**2**	

If the shipping cost is Rs.10 per load per mile, what shipping schedule should be used?

Solution

The given transportation problem is a balanced transportation problem, since,

$\sum a_i = \sum b_j = 8$. Hence, we find IBFS using VAM.

The difference between smallest and next smallest cost for each row and column is computed and written under penalty column and row.

	A	B	C	Availability	Unit Penalty
X	50	30	220	**1**	**(20)**
Y	90	45	170	**3**	**(45)**
Z	250	200	50	**4**	**(150)**
Requirement	3	3	2		
Unit Penalty	**(40)**	**(15)**	**(120)**		

The maximum penalty is 150 which belong to production center Z. We need to allocate in that row which has the least cost. Here, it is cell ZC. We allocate min (2, 4) = 2 units to cell ZC. Hence, the distribution at the center C is exhausted.

	A	B	C	Availability	Unit Penalty
X	50	30	20	**1**	**(20)**
Y	90	45	170	**3**	**(45)**
Z	250	200	50 $\boxed{2}$	**4** **= 2**	**(150)**
Requirement	**3**	**3**	**–**		
Unit Penalty	**(40)**	**(15)**	**(120)**		

Penalties are computed afresh after deleting column C. The maximum penalty in the reduced table is 50 which belongs to production unit Z. Allocation is done at the minimum cost cell of that production center Z. Allocate min (2, 3) = 2 to the cell having minimum cost i.e. ZB. Then, we find the unit Z gets exhausted i.e, row Z is deleted.

	A	B	Availability	Unit Penalty
X	50	30	**1**	**(20)**
Y	90	45	**3**	**(45)**
Z	250	200 $\boxed{2}$	–	**(50)**
Requirement	**3**	**~~3~~ =1**		
Unit Penalty	**(40)**	**(15)**		

In the reduced table, penalties are computed and highest penalty is chosen. Here, it is 45 i.e., production center Y. Allocation is done at the minimum cost cell in that center i.e., min (1, 3) =1 to the cell YB.

	A	B	Availability	Unit Penalty
X	50	30	**1**	**(20)**
Y	90	45 $\boxed{1}$	**~~3~~ =2**	**(45)**
Requirement	**3**	–		
Unit Penalty	**(40)**	**(15)**		

After deleting the exhausted distribution center B, we have, the production center A having minimum cost at cell XA. Allocate min (3, 1) = 1 to this cell.

	A	Availability	–
X	50 $\boxed{1}$	–	**(20)**
Y	90	**2**	–
Requirement	**~~3~~ = 2**		
Unit Penalty	**(40)**		

We observe that there are 2 production units available at A and required units are 2 at Y. We allocate 2 units to cell YA.

	A	Availability
Y	90 $\boxed{2}$	**2**
Requirement	**2**	

Since, supply and demand at each production center and distribution center is exhausted, the initial solution is arrived at as shown in the table below.

Conclusion: the number of allocated cells are five which is equal to the required number ($m + n - 1$) = (3 + 3 –1 = 5). Thus, this solution is a non-degenerate solution.

Distribution / Production	A	B	C	Availability
X	50 $\boxed{1}$	30	220	**1**
Y	90 $\boxed{2}$	45 $\boxed{1}$	170	**3**
Z	250	200 $\boxed{2}$	50 $\boxed{2}$	**4**
Requirement	**3**	**3**	**2**	

The shipping schedule is

$X \to A, Y \to A, Y \to B, Z \to B, Z \to C$ which gives the number of units as $1\times 50 + 5\times 90 + 1\times 45 + 2\times 200 + 2\times 50 = 775$ miles.

Since, shipping cost is Rs.10 per load per mile, for the given shipping schedule transportation cost is 775×10 = Rs.7, 750.

Example 7

A dairy firm has three plants located in a state. The daily milk production at each plant is as follows:

Plant 1: 6 million liters,

Plant 2: 1 million liters,

Plant 3: 10 million liters.

Each day, the firm must fulfill the needs of its four distribution centers. Minimum requirement at each center is as follows:

Distribution center 1: 7 million liters,

Distribution center 2: 5 million liters,

Distribution center 3: 3 million liters,

Distribution center 4: 2 million liters.

Cost in hundreds of rupees of shipping one million liter from each plant to each distribution center is given in the following table.

Plant \ Distribution	D_1	D_2	D_3	D_4
P_1	2	3	11	7
P_2	1	0	6	1
P_3	5	8	15	9

Find the IBFS for the given problem by using

i. NWCM

ii. LCEM

iii. VAM.

Solution

Plant \ Distribution	D_1	D_2	D_3	D_4	Supply
P_1	2	3	11	7	**6**
P_2	1	0	6	1	**1**
P_3	5	8	15	9	**10**
Demand	**7**	**5**	**3**	**2**	**17**

The given transportation problem is a balanced transportation problem, since,

$\sum a_i = \sum b_j$ =17. Hence, we find IBFS using three methods.

1. **North-West Corner Method**

Plant \ Distribution	D_1	D_2	D_3	D_4	Supply
P_1	2 $\boxed{6}$	3	11	7	**6**
P_2	1 $\boxed{1}$	0	6	1	**1**
P_3	5	8 $\boxed{5}$	15 $\boxed{3}$	9 $\boxed{2}$	**10**
Demand	**7**	**5**	**3**	**2**	

i. Allocating min (6, 7) = 6 to the North-west corner cell (P_1, D_1) exhausts supply at P_1 and leaves 1 unit as unsatisfied demand at D_1.

ii. Next move to cell (P_2, D_1). Allocate min (1, 7) = 1 to that cell, exhausting supply at P_2 and satisfying demand at D_1 at the same time.

iii. Next move to cell (P_3, D_2). Since, supply at P_3, is equal to the demand at D_2, D_3 and D_4, therefore, allocate 5 units to cell (P_3, D_2), 3 units to cell (P_3, D_3) and 2 units to the cell (P_3, D_4).

It may be noted that the number of allocated cells (also called basic cells) are 5 which is one less than the required number ($m + n - 1$) = (3 + 4 – 1 = 6). Thus, this solution is degenerate solution.

The transportation cost associated with this solution is

$$2 \times 6 + 1 \times 1 + 8 \times 5 + 15 \times 3 + 9 \times 2 = \text{Rs.}11{,}600.$$

2. **Least Cost Entry Method**

Plant \ Distribution	D_1	D_2	D_3	D_4	Supply
P_1	2 $\boxed{6}$	3	11	7	**6**
P_2	1	0 $\boxed{1}$	6	1	**1**
P_3	5 $\boxed{1}$	8 $\boxed{4}$	15 $\boxed{3}$	9 $\boxed{2}$	**10**
Demand	**7**	**5**	**3**	**2**	**17**

i. The least unit cost is 0 in cell (P_2, D_2), therefore min (1, 5) = 1 unit can be allocated here. This exhausts the supply at plant P_2; therefore row 2 is crossed out.

ii. The next least unit cost is 2 in cell (P_1, D_1). Therefore, min (6, 7) = 6 units can be allocated here. This exhausts the supply at plant P_1; therefore row 1 is crossed out.

iii. Since the total supply at plant P_3 is now equal to the unsatisfied demands at all the four distribution centres, therefore, remaining possible allocations satisfying the supply and demand conditions are made in cells (P_3, D_1), (P_3, D_2), (P_3, D_3), and (P_3, D_4).

The number of allocated cells in this case are six which is equal to the required number ($m + n - 1$) = (3 + 4 – 1 = 6). Thus, this solution is non-degenerate solution.

The transportation cost associated with this solution is

$$2 \times 6 + 5 \times 1 + 8 \times 4 + 15 \times 3 + 9 \times 2 = \text{Rs.}11{,}200.$$

3. **Vogel's Approximation Method**

Plant \ Distribution	D_1	D_2	D_3	D_4	Supply	Row Penalty		
P_1	2 $\boxed{1}$	3 $\boxed{5}$	11	7	**6**	1	1	5
P_2	1	0	6	1 $\boxed{1}$	**1**	0	–	–
P_3	5 $\boxed{6}$	8	15 $\boxed{3}$	9 $\boxed{1}$	**10**	3	3	4
Demand	**7**	**5**	**3**	**2**				
Column Penalty	1	3	5	6↑				
	3	5↑	4	2				
	3	–	4	2				

The maximum penalty is at D_4 i.e., 6. Hence, min (2, 1) = 1 unit is allocated to the cell (P_2, D_4) having least cost.

The maximum penalty is at D_2 i.e., 5 for the reduced table. Hence, min (5, 6) = 5 unit is allocated to the cell (P_1, D_2) having least cost.

The maximum penalty is 5 corresponding to first row P_1. We allocate 1unit in the cell (P_1, D_1) with minimum cost 2.

Since the total supply at plant P_3 is now equal to the unsatisfied demand at all the four distribution centers, remaining possible allocations satisfying the supply and demand conditions are made in cells (P_3, D_1), (P_3, D_3), and (P_3, D_4).

The number of allocated cells is six which is equal to the required number ($m + n - 1$) = (3 + 4 – 1 = 6). Thus, this solution is a non-degenerate solution.

The transportation cost associated with this solution is

$$2 \times 1 + 3 \times 5 + 1 \times 1 + 5 \times 6 + 15 \times 3 + 9 \times 1 = \text{Rs.}10{,}200.$$

Conclusion

The transportation cost obtained by using

i. North-West Corner Method = Rs.11,600.

ii. Least Cost Entry Method = Rs. 11,200.

iii. Vogel's Approximation Method = Rs.10,200.

It can be observed, from the above transportation costs, that the Vogel's Approximation Method yields least transportation cost (close to optimum cost) among the three methods. Hence, it is advisable to use VAM for IBFS.

TEST FOR OPTIMALITY

Once an initial solution is obtained, next step is to check its optimality. An optimal solution is one where there is no other set of transportation routes (allocations) that will further reduce the total transportation cost. Thus, we have to evaluate each unoccupied cell (represents unoccupied route) in the transportation table in terms of an opportunity of reducing total transportation cost.

An unoccupied cell with the largest negative opportunity cost is selected to include in the new set of transportation routes (allocations). This is also known as an incoming variable. The outgoing variable in the current solution is the occupied cell (basic variable) in the unique closed path (loop) whose allocation will become zero first, as more units are allocated to the unoccupied cell with largest negative opportunity cost. Such an exchange reduces total transportation cost. The process is continued until there is no negative opportunity cost. That is, the current solution cannot be improved further. This is the optimal solution.

To find the optimum solution we use two methods namely,

i. Stepping Stone Method.

ii. Modified Distribution Method (MODI).

Stepping Stone Method

The stepping stone method is an interactive technique from moving an initial basic feasible solution to an optimum solution. In order to apply the Stepping Stone Method, to transportation problem, one rule about the number of shipping routes being used must be observed first. In this rule, the number of occupied routes must always be equal to one less than the sum of the number of rows plus the number of columns.

Procedure:

Step 1: Prepare transportation table with a given unit cost of transportation along with the rim requirements.

Step 2: Determine an IBFS using any method (preferably VAM).

Step 3: Evaluate all unoccupied cells for the effect of transferring one unit from an occupied cell to unoccupied cell. This transfer is made by forming a closed path (loop) that retains the supply and demand condition of the problem. The evaluation is conducted as follows:

a. Select an unoccupied cell to be evaluated.

b. Beginning with the selected unoccupied cell, trace a closed path (or loop) through atleast three occupied cells and finally returning back to the same occupied cell. The direction of the movement taken is immaterial because the result will be same in either case.

   In the closed path formulation of only right angle turn is allowed, and therefore skip all other cells which are not at the turning points.

c. At each corner of the closed path assign plus (+) and minus (–) sign alternatively, beginning with plus sign for the unoccupied cell to be evaluated. The '+' and '–' signs can be assigned either in clockwise or anti-clockwise direction.

d. Compute the net change in cost along the closed path by adding together the unit transportation costs associated with each of the cells traced in the closed path.

e. Repeat steps (a) to (d) until net change in cost has been calculated for all unoccupied cells.

Step 4: Check the sign of each of the net change in the unit transportation costs. If all net changes are positive (+) or zero, then optimum solution is attained.

Step 5: Select the unoccupied cell with most negative net change among all unoccupied cells. If two minus values are equal, select that one which will result in moving as many units as possible into the selected unoccupied cell with minimum cost.

Step 6: Assign as many units as possible to unoccupied cell satisfying rim conditions. The maximum number of units to be assigned is equal to the smaller circled number ignoring sign among the occupied cells with minus value in the closed path.

Step 7: Go to step 3 and repeat the procedure until all unoccupied cells are evaluated and the net change is positive or zero values.

Example 8

A company manufacturing air-coolers has two plants located at Bombay and Kolkata with a weekly capacity of 200 units and 100 units, respectively. The company supplies air-coolers to its 4 showrooms situated at Ranchi, Delhi, Lucknow and Kanpur which have a demand of 75, 100, 100 and 30 units respectively, the cost of transportation per unit (in Rs.) is shown in the following table :

	Ranchi	Delhi	Lucknow	Kanpur
Bombay	90	90	100	100
Kolkata	50	70	130	85

Plan the production program so as to minimize the total cost of transportation.

Solution

The given information is tabulated as below:

	Ranchi	Delhi	Lucknow	Kanpur	Capacity
Bombay	90	90	100	100	200
Kolkata	50	70	130	85	100
Demand	75	100	100	30	300 / 305

The given transportation problem is an unbalanced transportation problem, since, $\sum a_i \neq \sum b_j$ where $\sum a_i = 300$; $\sum b_j = 305$. So, it should be balanced by introducing Dummy row with zero cost. Then the balanced transportation table is as follows:

	Ranchi	Delhi	Lucknow	Kanpur	Capacity
Bombay	90	90	100	100	200
Kolkata	50	70	130	85	100
Dummy	0	0	0	0	5
Demand	75	100	100	30	305 / 305

Hence, we find the IBFS using VAM. The IBFS using VAM is obtained as given below:

	Ranchi	Delhi	Lucknow	Kanpur	Demand	Unit penalties		
Bombay	90	90 $\boxed{75}$	100 $\boxed{95}$	100 $\boxed{30}$	**200**	0	0	10
Kolkata	50 $\boxed{75}$	70 $\boxed{25}$	130	85	**100**	20	20	15
Dummy	0	0	0 $\boxed{5}$	0	**5**	0	–	–
Requires	**75**	**100**	**100**	30				
Unit Penalties	50	70	↑ 100	85				
	↑ 40	20	30	15				
	–	20	↑ 30	15				
	–	↑ 20	–	15				

Hence, the IBFS is as follows:

	Ranchi (P)	Delhi (D)	Lucknow (L)	Kanpur (K)	Demand
Bombay (B)	90	90 $\boxed{75}$	100 $\boxed{95}$	100 $\boxed{30}$	**200**
Kolkata (C)	50 $\boxed{75}$	70 $\boxed{25}$	130	85	**100**
Dummy (O)	0	0	0 $\boxed{5}$	0	**5**
Requires	**75**	**100**	**100**	30	

The number of allocated cells is six which is equal to the required number $(m + n - 1) = (4 + 3 - 1 = 6)$. Thus, this solution is a non-degenerate solution.

The transportation cost with this IBFS is

$$90 \times 75 + 100 \times 95 + 100 \times 30 + 50 \times 75 + 70 \times 25 + 0 \times 5 = \text{Rs.}24,750.$$

After obtaining the IBFS, we move towards optimality. Here, we find the optimal solution using the stepping stone method as follows:

We compute the improvement index for all the unoccupied cells (unallocated cells). Here, the unallocated cells are PB, CL, CK, OP, OD, and OK. Each improvement index is shown separately in the following tables.

Hence, the improvements are summarized as follows:

Unoccupied Cell	Closed Path	Improvement Index
PB	PB – PC + CD – BD	90 – 50 – 70 + 90 = 20
CL	CL – CD + BD – BL	130 – 70 + 90 – 100 = 50
CK	CK – CD + BD – BK	85 – 70 + 90 – 100 = 5
OP	OP – CP + CD – BD + BL – OL	0 – 50 + 70 – 90 + 100 – 0 = 30
OD	OD – BD + BL – OL	0 – 90 + 100 – 5 = 5
OK	OK – OL + BL – BK	0 – 5 + 100 – 100 = 5

Since all the improvement indices are ≥ 0, the current solution is optimal.

Conclusion

Total cost of optimal solution is given as follows:

Transportation	Quantity Transported	Unit Cost	Total Cost
Bombay $\rightarrow$ Delhi	75	90	6750
Bombay $\rightarrow$ Lucknow	95	100	9500
Bombay $\rightarrow$ Kanpur	30	100	3000
Kolkata $\rightarrow$ Ranchi	75	50	3750
Kolkata $\rightarrow$ Delhi	25	70	1750
Dummy $\rightarrow$ Lucknow	5	0	0
Minimum transportation cost = Rs.24,750			

Modified Distribution Method (MODI)

The MODI method allows us to compute improvement indices quickly for each unbiased cell without drawing all of the closed paths (loops). Because of this it can often provide considerable time savings over the stepping stone method for solving transportation problems. It is based on the concept of Duality. It is also known as *u-v* method.

MODI provide a new means of finding the unoccupied route with the largest negative improvement index. Once the largest index is identified, we are required to trace only one closed path, just as with the stepping stone approach, this path helps to determine the maximum number of units that can be shipped via the best unoccupied route.

The following steps are followed to determine the optimality:

Step 1: For an initial basic feasible solution with $m + n - 1$ occupied cells, calculate u_i and v_i for rows and columns. To start with, any one of u_i or v_i's is given the value zero. It is better to assign zero for a particular u_i or v_i where there are maximum number of allocations in a row or column respectively, as it will considerably reduce arithmetic work. By using the relation $c_{ij} = u_i + v_j$ for all occupied cells (i, j), the computation of u_i's and v_i's for other rows and columns can be completed.

Step 2: For unoccupied cells, calculate opportunity cost by using the relation:

$$d_{ij} = c_{ij} - (u_i + v_j) \forall\, i, j$$

i.e., Opportunity cost = Actual cost – Implied cost.

Step 3: Examine sign of each d_{ij} .

i. If all $d_{ij} > 0$, then the current basic solution is optimum.

ii. If at least one of the $d_{ij} = 0$, then there exists an alternate optimum solution.

iii. If at least one of the $d_{ij} < 0$, then an improved solution can be obtained by entering the unoccupied cell (i, j) in the basis. An unoccupied cell having the largest negative value of d_{ij} is chosen for entering into the new transportation schedule.

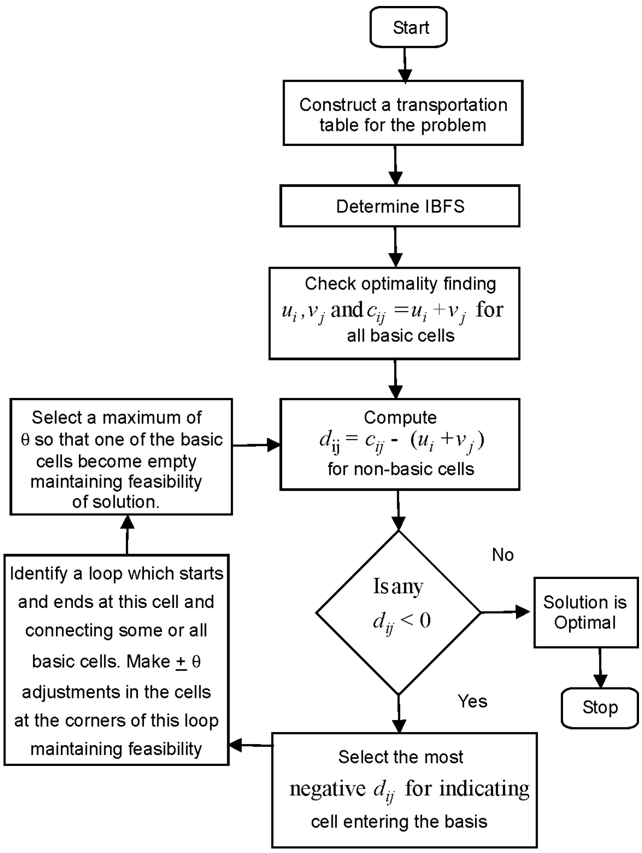
Step 4: Construct a closed path (or loop) for the unoccupied cell with largest negative opportunity cost. Start the close loop with the selected unoccupied cell and mark a plus sign (+) in this cell, trace a path along the rows (or columns) to an occupied cell, mark the corner with minus sign (–) and continue down the column (or row) to an occupied cell and mark the corner with alternatively plus sign (+) and minus sign (–). Close the path back to the selected unoccupied cell.

Step 5: Locate the smallest quantity allocated to a cell marked with a minus sign. Allocate this value to the selected unoccupied cell and add it to other occupied cells marked with plus signs and subtract it from the occupied cells marked with minus signs.

Step 6: Obtain a new improved solution after allocating units to the unoccupied cell according to step 5 and calculate the new total transportation cost.

Step 7: Test the revised solution for optimality.

Figure: Flow Chart for MODI Method



Example 9

The following table shows all the necessary information on the available supply to each warehouse, the requirement of each market and the unit transportation cost from each warehouse to each market.

		Market				Supply
		P	Q	R	S	
Warehouse	X	5	2	4	3	**22**
	Y	4	8	1	6	**15**
	Z	4	6	7	5	**8**
	Demand	7	**12**	**17**	**9**	

The shipping clerk has worked-out the following schedule from experience: 12 units from X to Q, 1 unit from X to R, 9 units from X to S, 15 units from Y to R, 7 units from Z to P and 1 unit from Z to R.

a. Check and see if the clerk has the optimum schedule.

b. Find the optimal schedule and minimum total shipping cost.

Solution

The given transportation problem is a balanced transportation problem, since, $\sum a_i = \sum b_j$ = 45. Hence we find IBFS using LCEM.

The following schedule is worked-out by the chek:

	P	Q	R	S	Supply
X	5	2 $\boxed{12}$	4 $\boxed{1}$	3 $\boxed{9}$	**22**
Y	4	8	1 $\boxed{15}$	6	**15**
Z	4 $\boxed{7}$	6	7 $\boxed{1}$	5	**8**
Demand	**7**	**12**	**17**	**9**	**45**

The number of allocated cells are six which is equal to the required number (m + n – 1) = (3 + 4 – 1 = 6). Thus, this solution is a non-degenerate solution.

Hence, we move towards optimality using MODI method.

We compute u_i and v_j using the following relationship:

$$c_{ij} = u_i + v_j \;\; \forall i, j .$$

Let $u_1 = 0$ and calculate u_i's and v_j's for i = 1,2,3 and j = 1,2,3,4 using the allocated cells.

We have $c_{ij} = u_i + v_j \;\forall i, j$ which implies

$c_{12} = u_1 + v_2 \Rightarrow 2 = 0 + v_2 \Rightarrow v_2 = 2$

$c_{13} = u_1 + v_3 \Rightarrow 4 = 0 + v_3 \Rightarrow v_3 = 4$

$c_{14} = u_1 + v_4 \Rightarrow 3 = 0 + v_4 \Rightarrow v_4 = 3$

$c_{23} = u_2 + v_3 \Rightarrow 1 = u_2 + 4 \Rightarrow u_2 = -3$

$c_{33} = u_3 + v_3 \Rightarrow 7 = u_3 + 4 \Rightarrow u_3 = 3$

$c_{31} = u_3 + v_1 \Rightarrow 4 = 3 + v_1 \Rightarrow v_1 = 1.$

Hence, we have $u_1 = 0$, $u_2 = -3$, $u_3 = 3$ and $v_1 = 1$, $v_2 = 2$, $v_3 = 4$, $v_4 = 3$.

The table with the calculated u_i 's and v_j 's is as follows:

		$v_1 = 1$	$v_2 = 2$	$v_3 = 4$	$v_4 = 3$	
		P	Q	R	S	Supply
$u_1 = 0$	X	5	2 $\boxed{12}$	4 $\boxed{1}$	3 $\boxed{9}$	**22**
$u_2 = -3$	Y	4	8	1 $\boxed{15}$	6	**15**
$u_3 = 3$	Z	4 $\boxed{7}$	6	7 $\boxed{1}$	5	**8**
	Demand	**7**	**12**	**17**	**9**	**45**

With the calculated values of u_i 's and v_j 's we compute d_{ij} 's using the relation,

$d_{ij} = c_{ij} - (u_i + v_j)\ \ \forall\, i, j$ for all unallocated cells.

$d_{11} = c_{11} - (u_1 + v_1) = 5 - (0+1) \Rightarrow d_{11} = 4$

$d_{21} = c_{21} - (u_2 + v_1) = 4 - (-3+1) \Rightarrow d_{22} = 6$

$d_{22} = c_{22} - (u_2 + v_2) = 8 - (-3+2) \Rightarrow d_{22} = 9$

$d_{24} = c_{24} - (u_2 + v_4) = 6 - (-3+3) \Rightarrow d_{24} = 6$

$d_{32} = c_{32} - (u_3 + v_2) = 6 - (3+2) \Rightarrow d_{32} = 1$

$d_{34} = c_{34} - (u_3 + v_4) = 5 - (3+3) \Rightarrow d_{34} = -1$

Here, we find one $d_{ij} < 0$. Hence, the solution is not optimal.

Since the most negative d_{ij} is (–1) in the cell ZS, the closed path for ZS is shown below:

XR (1 + θ) $\boxed{1}$	XS (9 – θ) $\boxed{9}$
ZR (1 – θ) $\boxed{1}$	ZS (θ)

θ is allotted in ZS, and –θ, and +θ adjustments have been done to the corner allocations of the loop since θ is subtracted from 9 and 1. Therefore θ=

XR 1 + 1= 2 $\boxed{2}$	XS 9–1 = 8 $\boxed{8}$
ZR 1 – 1= 0	ZS 0+1 = 1 $\boxed{1}$

Now the improved solution is

	P	Q	R	S	Supply
X	5	2 $\boxed{12}$	4 $\boxed{2}$	3 $\boxed{8}$	**22**
Y	4	8	1 $\boxed{15}$	6	**15**
Z	4 $\boxed{7}$	6	7	5 $\boxed{1}$	**8**
Demand	**7**	**12**	**17**	**9**	

To find whether the improved solution is optimal or not, we calculate u_i 's and v_j 's and hence, d_{ij} 's for the above table as follows:

Compute u_i 's and v_j 's for the allocated cells and d_{ij} 's for non-allocated cells as given below:

The table with the calculated u_i 's and v_j 's is as follows:

		$v_1=2$	$v_2=2$	$v_3=4$	$v_4=3$	
		P	Q	R	S	Supply
$u_1=0$	X	5	2 $\boxed{12}$	4 $\boxed{2}$	3 $\boxed{8}$	**22**
$u_2=-3$	Y	4	8	1 $\boxed{15}$	6	**15**
$u_3=2$	Z	4 $\boxed{7}$	6	7	5 $\boxed{1}$	**8**
	Demand	**7**	**12**	**17**	**9**	**45**

With the calculated values of u_i 's and v_j 's we compute d_{ij} 's using the relation,

$d_{ij} = c_{ij} - (u_i + v_j) \forall i, j$ for all unallocated cells.

$d_{11} = c_{11} - (u_1 + v_1) = 5 - (0 + 2) \Rightarrow d_{11} = 3$

$d_{21} = c_{21} - (u_2 + v_1) = 4 - (-3 + 2) \Rightarrow d_{22} = 5$

$d_{22} = c_{22} - (u_2 + v_2) = 8 - (-3 + 2) \Rightarrow d_{22} = 9$

$d_{24} = c_{24} - (u_2 + v_4) = 6 - (-3 + 3) \Rightarrow d_{24} = 6$

$d_{32} = c_{32} - (u_3 + v_2) = 6 - (2 + 2) \Rightarrow d_{32} = 2$

$d_{33} = c_{33} - (u_3 + v_3) = 7 - (2 + 4) \Rightarrow d_{34} = 1$

Since all the d_{ij}'s ≥ 0, the current solution is optimal.

Conclusion

a. The clerk does not have the optimum schedule. Total cost with his schedule is Rs.105.

b. The optimum schedule is

From Market	To Warehouse	Quantity shipped
X	Q	12
X	R	2
X	S	8
Y	R	15
Z	P	7
Z	S	1

The minimum transportation cost is
$12\times2+2\times4+8\times3+15\times1+7\times4+1\times5=\text{Rs. }104$.

PROFIT MAXIMIZATION IN TRANSPORTATION PROBLEM

A method of solving maximization case would be to convert the problem into a minimization case. The method is to select the largest element from the profit pay-off matrix and then subtracting all the elements from this largest element including itself. The reduced matrix so obtained becomes the minimization case and it can be solved by any one of the methods explained earlier i.e., NWCM, LCEM, VAM for IBFS and Stepping Stone or MODI for optimum solution.

Example 10

A company has four factories A, B, C, D manufacturing the same product. Production and raw material costs differ from factory to factory and are given in the following table in the first two rows. The transportation costs from the factories to sales depots P, Q, R are also given. The last two columns in the table give the sales price and total requirement at each depot. The production capacity of each factory is given in the last row.

		A	B	C	D	Sales per unit	Requirement
Production cost/unit		15	18	14	13		
Raw material cost/unit		10	9	12	9		
Transportation cost/unit	P	3	9	5	4	34	80
	Q	1	7	4	5	32	120
	R	5	8	3	6	31	150
Production Capacity		10	150	50	100		

Determine the most profitable production and distribution schedule and the corresponding profit. The surplus production should be taken to yield zero profit.

Solution

The profit matrix as given in the problem can be constructed by using the following equation:

Profit = Sales price – Production cost – Raw material cost – Transportation cost.

From \ To	P	Q	R	Capacity
A	6	6	1	10
B	–	–2	–4	150
C	3	2	2	50
D	8	5	3	100
Requirement	80	120	150	310 / 350

Since the total requirement exceeds total capacity by 40 units, the given problem is an unbalanced problem. We, therefore, introduce a dummy factory E with its capacity of 40 units and transportation costs from this location to all sales depots is zero as shown in the following table:

From \ To	P	Q	R	Capacity
A	6	6	1	10
B	–2	–2	–4	150
C	3	2	2	50
D	8	5	3	100
E	0	0	0	40
Requirement	80	120	150	350 / 350

To convert this balanced TP of maximization case to minimization case, select the largest element (i.e., 8) from the table and then subtract each element of the above transportation profit table from this largest element. The relative loss matrix so obtained is shown in the following table:

From \ To	P	Q	R	Capacity
A	2	2	7	10
B	10	10	12	150
C	5	6	6	50
D	0	3	5	100
E	8	8	8	40
Requirement	80	120	150	350 / 350

The IBFS for the given problem obtained by using VAM is as follows:

	P	Q	R	Capacity
A	2	2 [10]	7	**10**
B	10	10 [40]	12 [110]	**150**
C	5	6 [50]	6	**50**
D	0 [80]	3 [20]	5	**100**
E	8	8	8 [40]	**40**
Available	**80**	**120**	**150**	

The number of allocated cells are seven, equal to the required number $(m + n - 1)$ = (3 + 5 – 1 = 7). Thus, this solution is a non-degenerate solution.

After obtaining the initial solution, we move towards optimality using MODI method.

We compute u_i and v_j using the following relationship:

$$c_{ij} = u_i + v_j \ \forall i, j .$$

Let $u_1 = 0$ and calculate u_i's and v_j's for i =1,2,3 and j =1,2,3,4 using the allocated cells and with the calculated values of u_i's and v_j's.

Where, $u_2 = 8, u_3 = 2, u_4 = -2, u_5 = -4$ and $v_1 = 2, v_2 = 2, v_3 = 4$.

We compute d_{ij}'s using the relation,

$d_{ij} = c_{ij} - (u_i + v_j) \forall i, j$ for all unallocated cells

		$v_1 = -3$	$v_2 = 0$	$v_3 = 2$	Capacity
		P	Q	R	
$u_1 = 2$	A	2 **(3)**	2 $\boxed{10}$	7 **(3)**	**10**
$u_2 = 10$	B	10 **(3)**	**(+)**10 $\boxed{40}$	**(–)**12 $\boxed{110}$	**150**
$u_3 = 6$	C	5 **(2)**	**(–)**6 $\boxed{50}$	**(+)**6 **(–2)**	**50**
$u_4 = 3$	D	0 $\boxed{80}$	3 $\boxed{20}$	5 **(0)**	**100**
$u_5 = 6$	E	8 **(5)**	8 **(2)**	8 $\boxed{40}$	**40**
	Available	**80**	**120**	**150**	

Most negative net evaluation is (–2) in the CR cell, we form a loop and get new solution.

		$v_1 = -3$	$v_2 = 0$	$v_3 = 2$	Capacity
		P	Q	R	
$u_1 = 2$	A	2 **(3)**	2 $\boxed{10}$	7 **(3)**	**10**
$u_2 = 10$	B	10 **(3)**	10 $\boxed{90}$	12 $\boxed{60}$	**150**
$u_3 = 6$	C	5 **(4)**	6 **(2)**	6 $\boxed{50}$	**50**
$u_4 = 4$	D	0 $\boxed{80}$	3 $\boxed{20}$	5 **(0)**	**100**
$u_5 = 6$	E	8 **(5)**	8 **(2)**	8 $\boxed{40}$	**40**
	Available	**80**	**120**	**150**	

It is found that all the cell evaluations of d_{ij}'s ≥ 0 . Therefore, the current solution is optimal.

Conclusion

Transport	Units
$A \to Q$	10×1
$B \to Q$	90×7
$B \to R$	60×8
$C \to R$	50×3
$D \to P$	80×4
$D \to Q$	20×5
$E \to R$	40×0

The maximum profit (using the costs/unit given the original table) = Rs.1, 690.

TIME-MINIMIZING TRANSPORTATION PROBLEMS

In time-minimizing transportation problems, the objective is to minimize the time of transportation rather than the cost of transportation. For example in military, the time of supply is considered more valuable than the cost of transportation and therefore, it is always preferred to minimize the total time of supply and not the cost. Such problems are often encountered in emergency services like military services, hospital management, fire services, etc.

The time-minimization transportation problems are similar to the cost-minimization transportation problems, except that the unit transportation cost c_{ij} is replaced by the unit time t_{ij} required to transport the items from *i*th origin to *j*th destination.

The feasible plan (IBFS) for this problem can also be obtained by using any one of the methods discussed earlier. Now, if T_j is the total transportation time associated with the *j*th feasible solution, then we have to find such solution which gives minimum (T_j).

The procedure is given below:

Step 1: First, find an IBFS by using any one of the methods discussed. Enter the solution at the centers of the basic cells.

Step 2: Compute T_j for this BFS and cross out all the non-basic cells for which $t_{ij} \geq T_j$.

Step 3: Now construct a loop for the basic cells corresponding to T_j in such a way that when the values at the center of the cells are shifted around, the value at this cell tends towards (not-necessarily) zero and no variable becomes zero. If no such closed path is possible, the solution under test is optimal, otherwise go to step 2.

Step 4: Repeat the procedure until an optimum basic feasible solution is attained.

Example 11

If the matrix elements represent the time taken to transport units from three sources to four destinations to find the minimum transportation time, solve the following TP:

	D_1	D_2	D_3	D_4	Available
S_1	10	0	20	11	15
S_2	1	7	9	20	25
S_3	12	14	16	18	5
Required	12	8	15	10	45

Solution

First Iteration

Step 1: Using VAM, find an IBFS. The IBFS is obtained as follows:

	D_1	D_2	D_3	D_4	Available
S_1	10	0 $\boxed{8}$	20	11 $\boxed{7}$	15
S_2	1 $\boxed{12}$	7	9 $\boxed{13}$	20	25
S_3	12	14	16 $\boxed{2}$	18 $\boxed{3}$	5
Required	12	8	15	10	45

Step 2: The times for this feasible plan are:

$t_{12} = 0,\ t_{14} = 11,\ t_{21} = 1,\ t_{23} = 9,\ t_{33} = 16$ and $t_{34} = 18$.

Therefore, $T = \max[0,11,1,9,16,18] = 18$.

Obviously, all the shipments for this plan are to be completed within the time $T_1 = 18$. So, we cross-out the cell (1, 3) and (2, 4) because $t_{13} > T_1$ and $t_{24} > T_1$.

	D_1	D_2	D_3	D_4
S_1	10	0 (–) $\boxed{8}$	~~20~~	11 (+) $\boxed{7}$
S_2	1 $\boxed{12}$	7	9 $\boxed{13}$	~~20~~
S_3	12	14 (+)	16 $\boxed{2}$	18 (–) $\boxed{3}$

Step 3: Find the closed path (loop) for the cell (3, 4) for which $t_{34} = T_1 = 18$. It is clear from the above table that only 3 units can be shifted around.

Second Iteration

Step 1: The revised feasible plan is as given below:

	D_1	D_2	D_3	D_4
S_1	10	0 $\boxed{5}$	20	11 $\boxed{10}$
S_2	1 (–) $\boxed{12}$	7	9 (+) $\boxed{13}$	20
S_3	12 (+)	14 $\boxed{3}$	16 (–) $\boxed{2}$	18

Step 2: Here, $T_2 = \max[t_{12}, t_{14}, t_{21}, t_{22}, t_{32}, t_{33}] = \max[0,11,1,9,14,16] = 16$.

As all shipments for this feasible solution are completed within time $T_2 = 16$, we cross-out the cell (3, 4) also since $t_{34} > T_2$.

Step 3: The closed loop starts from cell (3, 3) as shown in the table below: It is clear from the table that only 2 units can be shifted around.

Third Iteration

Step 1: The revised plan is as shown in the table below:

	D_1	D_2	D_3	D_4
S_1	10	0 $\boxed{5}$	$\cancel{20}$	11 $\boxed{10}$
S_2	1 (–) $\boxed{10}$	7 (+)	9 $\boxed{15}$	$\cancel{20}$
S_3	12 (+) $\boxed{2}$	14 (–) $\boxed{3}$	$\cancel{16}$	$\cancel{18}$

Step 2: Here $T_3 = \max\left[t_{12}, t_{14}, t_{21}, t_{23}, t_{31}, t_{32}\right] = \max\left[0, 11, 1, 9, 12, 14\right] = 14$.

As all shipments for this feasible solution are completed within time $T_3 = 14$, we cross-out the cell (3, 3) also since $t_{33} > T_3$.

Step 3: Since $t_{32} = T_3 = 14$, the closed loop starts from the cell (3, 2) as shown in the table below: It is clear that from the table that only 3 units can be shifted around.

Fourth Iteration

Step 1: The revised plan is as shown in the table below:

	D_1	D_2	D_3	D_4
S_1	10	0 $\boxed{5}$	20	11 $\boxed{10}$
S_2	1 $\boxed{7}$	7 $\boxed{3}$	9 $\boxed{15}$	$\cancel{20}$
S_3	12 $\boxed{5}$	$\cancel{14}$	$\cancel{16}$	$\cancel{18}$

Step 2: Here $T_4 = \max\left[t_{12}, t_{14}, t_{21}, t_{22}, t_{23}, t_{31}\right] = \max\left[0, 11, 1, 7, 9, 12\right] = 12$.

As all shipments for this feasible solution are completed within time $T_4 = 12$, we cross-out the cell (3, 2) also since $t_{32} > T_4$.

Step 3: Now we cannot form any closed loop without increasing the present minimum shipping time. Hence, the feasible plan at this stage is optimal.

Conclusion

Thus, all the shipments can be made within 12 time units with the following transportation schedule.

Transport	Units
$S_1 \to D_2$	5
$S_1 \to D_4$	10
$S_2 \to D_1$	7
$S_2 \to D_2$	3
$S_2 \to D_3$	15
$S_3 \to D_1$	5

TRANSSHIPMENT PROBLEM

A transportation problem in which available commodity may not be dispatched directly from the source to the destination, i.e., it passes through one or more sources and destinations before reaching the final destination is termed as transshipment problem. Such a problem as such cannot be solved by the transportation technique, but slight modification enables us to work-out with usual transportation technique.

There are several situations where it might be economical or convenient or necessary to transport items in more than one stages. Movement of material involving two different modes of transport, road and railways or broad gauge system and meter gauge system both will require transshipment from one mode of transportation to another at the change point. Thus, it will be of practical interest to find the minimum cost transportation schedule without ignoring the possibility or the need of transshipment.

A transshipment problem can be changed into a standard transportation problem by creating the source and supply points as both sources as well as supply points with the addition of some quantity of buffer stock required at sources for supply points and at supply points for sources.

Definition

A transportation problem in which available commodity frequently moves from one source to another source or destination before reaching its actual destination is called a trans-shipment problem.

MAIN CHARACTERISTICS OF TRANSSHIPMENT PROBLEM

Following are the main characteristics of transshipment problems:

i. The number of sources and destinations in the transportation problem are m and n respectively. But in the transshipment problems, we have $m + n$ sources and destinations.

ii. If S_i denotes the ith source and D_j denotes the jth destination, then commodity can move along the route $S_i \to D_i \to D_j$, $S_i \to S_j \to D_i \to D_j$, $S_i \to D_i \to S_j \to D_j$, or in various other ways. Clearly, transportation cost from S_i to S_i is zero and transportation cost from S_i to S_j or S_i to D_i do not have to be symmetrical, i.e., in general, $S_i \to S_j \neq S_j \to S_i$.

iii. While solving the transshipment problem, first we change it into a standard transportation problem, and then proceed in the same manner as in solving the transportation problems.

iv. The basic feasible solution contains $2m + 2n - 1$ basic variables. If we omit the variables appearing in $(m + n)$ diagonal cells, we are left with $m + n - 1$ basic variables.

Example 12

Consider the following transshipment problem with two sources and two destinations, the costs for shipment in rupees are given below. Determine the shipping schedule:

	S_1	S_2	D_1	D_2	
S_1	0	1	3	4	5
S_2	1	0	2	4	25
D_1	3	2	0	1	
D_2	4	4	1	0	
			20	10	30

Solution

***Step* 1: To get modified transportation problem**

In the transshipment problem, each given source and destination can be considered as a source or destination. If we now take the quantity available at each of the sources D_1 and D_2 to be zero and also at each of the destinations S_1 and S_2 the requirement to be zero, then the supply and demand from all the points (sources or destinations) a fictitious supply and demand quantity termed as 'bufferstock' is assumed and is added to both supply and demand of all the points. Generally, this buffer stock is chosen equal to $\sum a_i$ or $\sum b_j$. In our problem, the bufferstock comes out to be 30 units.

	S_1	S_2	D_1	D_2	Available
S_1	0	1	3	4	35
S_2	1	0	2	4	55
D_1	3	2	0	1	30
D_2	4	4	1	0	30
Required	30	30	50	40	

***Step* 2: To find initial solution of the modified problem**

By adding 30 units of commodity to each point of supply and demand, an IBFS is obtained by using VAM, as given below:

	S_1	S_2	D_1	D_2	Available
S_1	0 [30]	1	3	4 [5]	35
S_2	1	0 [30]	2 [20]	4 [5]	55
D_1	3	2	0 [30]	1	30
D_2	4	4	1	0 [30]	30
Required	30	30	50	40	30

***Step* 3: To apply Optimality test**

The variables $u_i\left(i=1,2,3,4\right)$ and $v_j\left(j=1,2,3,4\right)$ have been determined by using successively the relations $u_i+v_j=c_{ij}$ for all the basic (occupied) cells starting with $v_2=0$. These values are then used to compute the net evaluations $d_{ij}=c_{ij}-\left(u_i+v_j\right)$ for all the non-basic (empty) cells.

Clearly, d_{34} (= –1) is the only negative quantity. Hence assign +ve sign to this cell. After identifying the loop, we find that minimum among the negative d_{ij} 's is 5. This makes the cell (2, 4) leaves the basis (i.e., becomes empty).

		$u_1 = 0$	$u_2 = 0$	$u_3 = -2$	$u_4 = -4$	
		S_1	S_2	D_1	D_2	Available
$v_1 = 0$	S_1	0 $\boxed{30}$	1	3	4 $\boxed{5}$	35
$v_2 = 0$	S_2	1	0 $\boxed{30}$	2 (+) $\boxed{20}$	4 (–) $\boxed{5}$	55
$v_3 = 2$	D_1	3	2	0 (–) $\boxed{30}$	1 (+)	30
$v_4 = 4$	D_2	4	4	1	0 $\boxed{30}$	30
	Required	30	30	50	40	30

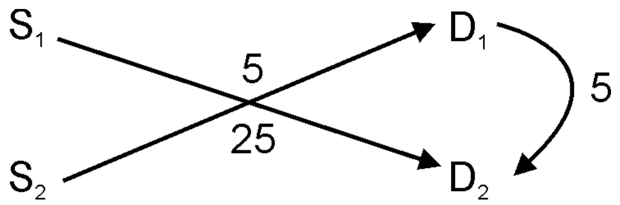
***Step* 4: Introduce the cell (3, 4) into the basis and drop the cell (2, 4) from the basis. Then again test the optimality of the revised solution.**

		$u_1 = 0$	$u_2 = -1$	$u_3 = -3$	$u_4 = -4$	
		S_1	S_2	D_1	D_2	Available
$v_1 = 0$	S_1	0 $\boxed{30}$	1	3	4 $\boxed{5}$	35
$v_2 = 1$	S_2	1	0 $\boxed{30}$	2 $\boxed{25}$	4	55
$v_3 = 3$	D_1	3	2	0 $\boxed{25}$	1 $\boxed{5}$	30
$v_4 = 4$	D_2	4	4	1	0 $\boxed{30}$	30
	Required	30	30	50	40	30

Since all the current net evaluations are non-negative, the current solution is an optimum one.

Conclusion: The minimum transportation cost is = $5 \times 4 + 25 \times 2 + 5 \times 1 = 75$

and the optimum transportation route is as shown below:



SUMMARY

- Transportation problem is a special case of LPP.
- Transportation technique helps to determine the least cost route of transportation of goods from company's different plants to different warehouses.
- Basic requirement of the problem is that total capacity and demand must be perfectly synchronized.

- Three basic methods at the initial solutions are North-West Corner Method, Least Cost Entry Method and Vogel's Approximation Method.
- Stepping Stone method and Modified Distribution method are two well known methods to improve the initial solution.
- The TP is said to be non-degenerate if the number of allocated cells is equal to $(m + n - 1)$.
- A TP which passes through one or more sources and destinations before reaching the final destination is termed as trans-shipment problem.

Exercise

1. Give an algorithm for solving TP.
2. State TP. Describe clearly the steps involved in solving the problem.
3. Describe the TP. Give a method of finding an IBFS. What is meant by an optimality test? Give the method of improving over the initial solution to reach the optimal feasible solution.
4. Describe the computational procedure of optimality test in a TP.
5. How do you diagnosis that a given TP is having more than one optimal solution?
6. Indicate how transshipment problem can be solved as a transportation problem.
7. M sources have s_i , i = 1,2,…,M units of commodity, while demand at retail shop j is d_j , j = 1,2,…, N. Profit per unit supplied from source i to shop j is p_{ij} , i =1,2,…,M; j = 1,2,…, N. Indicate how the profit-maximizing problem with the above data can be converted to an equivalent cost-minimizing problem, stating clearly any more assumptions one can make in this regard.
8. What is meant by balanced TP? Give a method of solving TP with capacities.
9. Explain in detail, any one method for solving a TP. Would you recommend this method to solve an assignment problem?
10. What is degeneracy in TP? How is TP solved when demand and supply are not equal?
11. Explain in brief, with examples: (i) North-West Corner Method, and (ii) Vogel's Approximation Method.
12. Distinguish between Time minimization and Cost minimization transportation problems.
13. Explain TP and show that it can be considered as LPP.
14. What are the common methods to obtain an IBFS for a TP whose cost and requirement table is given? Give a stepwise procedure for one of them.
15. What is meant by optimality test in TP?
16. Find the optimum solution to the following TP for which the cost, origin-availabilities, and destination-requirements are as given below:

From \ To	A	B	C	D	E	a_i
I	3	4	6	8	8	20
II	2	10	1	5	30	30
III	7	11	20	40	15	15
IV	2	1	9	14	18	13
b_j	40	6	8	18	6	

17. Find the optimum solution to the following TP:

To \ From	1	2	3	4	Supply
1	5	3	6	4	30
2	3	4	7	8	15
3	9	6	5	8	15
Demand	10	25	18	7	60

18. Goods have to be transported from factories F_1, F_2 and F_3 to warehouses W_1, W_2 W_3 and W_4. The transportation cost per unit, capacities and requirement of the warehouse are given in the following table:

From \ To	W_1	W_2	W_3	W_4	Capacity
F_1	27	23	31	69	150
F_2	10	45	40	32	40
F_3	30	54	35	57	80
Requirement	90	70	50	60	

Find the distribution of goods.

19. Solve the following TP:

From \ To	D_1	D_2	D_3	D_4	Supply
O_1	1	2	–2	3	70
O_2	2	4	0	1	38
O_3	1	2	–2	5	32
Demand	40	28	30	32	

20. ABC limited has three production shops supplying a product of five warehouses. The cost of production varies from shop to shop and cost of transportation from one shop to a warehouse also varies. Each shop has a specific production capacity and each warehouse has certain amount of requirement. The cost of transportation is as given below:

Shop	Warehouse					Capacity
	I	II	III	IV	V	
A	6	4	4	7	5	100
B	5	6	7	4	8	125
C	3	4	6	3	4	175
Demand	60	80	85	105	70	400

Find the minimum cost of manufacture of the product to be supplied from each shop to different warehouses.

21. Determine the optimum solution to each of the following degenerate TP:

i.

From \ To	D_1	D_2	D_3	D_4	D_5	a_i
O_1	4	7	3	8	2	4
O_2	1	4	7	3	8	7
O_3	7	2	4	7	7	9
O_4	4	8	2	4	7	2
b_j	8	3	7	2	2	

ii.

To / From	D_1	D_2	D_3	D_4	Supply
S_1	2	3	11	7	6
S_2	1	0	6	1	1
S_3	5	8	15	10	10
Demand	7	5	3	2	17

22. A company has three factories I, II, III and four warehouses 1,2,3,4. The transportation cost (in Rs.) per unit from each factory to each warehouse, the requirements of each warehouse, and the capacity of each factory are given below:
Find the minimum transportation schedule.

	1	2	3	4	Capacity
I	25	17	25	14	300
II	15	10	18	24	500
III	16	20	8	13	600
Required	300	300	500	500	

23. Solve the following TP, the matrix represents the time t_{ij}.

To / From	D_1	D_2	D_3	D_4	Available
O_1	6	7	3	4	5
O_2	7	9	1	2	7
O_3	6	5	16	7	8
O_4	18	9	10	2	10
Required	10	5	10	5	30

24. The following information is available concerning the operation of the XYZ manufacturing company:

Period	Units on order	Production Capacity		Excess cost per unit over time (Rs.)	Storage cost per unit (Rs.)
		Regular time	Over time		
Month 1	800	920	920	1.25	0.50
Month 2	1400	250	250	1.25	0.50

Formulate the above problem as a TP.

Chapter VII

Assignment Problem

After reading this chapter, you will be conversant with:

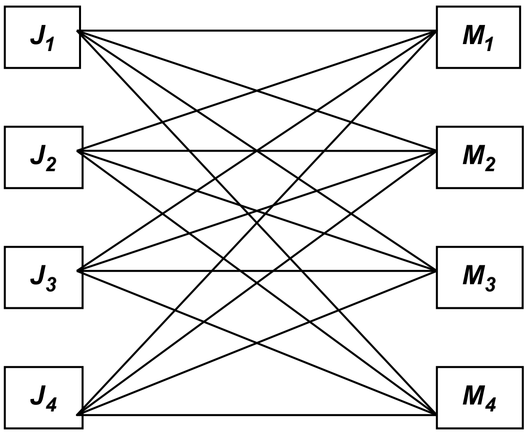
- Assignment Model as LPP
- Solving Assignment Problems
- Maximization Case in Assignment Problem
- Traveling Salesman Problem
- Unbalanced Assignment Problem
- Crew Assignment Problem

Introduction

Assignment Problem (AP) is a special type of LPP. It deals with allocating the various resources or items to various activities on one to one basis in such a way that the time or cost involved is minimized and sale or profit is maximized. Such type of problems can be solved with the help of simplex method or by transportation method but a simpler and more efficient method for getting the solution is through assignment technique.

Assignment technique is a typical optimization technique, particularly useful in a situation where a certain number of tasks require to be assigned to equal number of facilities, one task to each facility. The tasks differ in their work contents and the facilities differ in their capabilities. Typical examples of decision-making situations where assignment technique can be successfully used include assignments of:

- Jobs to machines.
- Sales personnel to sale territories.
- Vehicles to routes.
- Buildings to sites.
- Products to factories.
- Contracts to bidders.
- Project engineers to projects.



Small size assignment problems can be solved by enumerating and evaluating all combinations and selecting the optimal one. However, this becomes difficult as problems become bigger. For example, for $n \times n$ problem, there are $n!$ solutions and even a small 6×6 problem would have $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ possible solutions. To evaluate each of the combinations will consume hours and becomes an exercise in futility.

To solve such problems, assignment algorithm was developed by D Konig, a Hungarian mathematician. Therefore, assignment techniques are also known as Hungarian method of assignment problems.

Several problems of management has a structure identical with the assignment problem. Some of these are as follows:

- A departmental head may have six people available for assignment and six jobs to assign. He may like to know which job should be assigned to which person so that all these jobs can be completed in the shortest possible time.
- A truck company, may have an empty truck in each of cities 1,2,3,4,5,6 and needs an empty truck in each of the cities A,B,C,D,E,F. He would like to ascertain the assignment of trucks to various cities so as to minimize the total distance covered.

- In a marketing set up by making an estimate of sales performance for different territories one could assign a particular salesman a particular territory with a view to maximize overall sales.

Suppose that there are '*n*' jobs to be performed and '*n*' persons are available for doing these jobs. Assume that each person can do each job at a time, though with varying degrees of efficiency.

Let C_{ij} be the cost (payment) if the *i*th person is assigned *j*th job, the problem is to find an assignment (which job should be assigned to which person) so that the total cost of performing all jobs is minimum.

The assignment problem is constructed as follows:

Person	Jobs					Total
	1	2	3	...*j*...	*n*	1
1	C_{11}	C_{12}	C_{13}	C_{1j}	C_{1n}	1
2	C_{21}	C_{22}	C_{23}	C_{2j}	C_{2n}	1
3	C_{31}	C_{32}	C_{33}	C_{3j}	C_{3n}	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	C_{i1}	C_{i2}	C_{i3}	$\mathbf{C_{ij}}$	C_{in}	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	C_{n1}	C_{n2}	C_{n3}	C_{nj}	1	1
Total	1	1	1	$\cdots$	1	*n*

It is a special case of transportation problem in which persons represent sources, and jobs represent the destinations i.e., number of rows is equal to number of columns in the ($m = n$) assignment matrix and the supply amount at each source and demand amount at each destination exactly equal to one i.e., $a_i = b_j = 1$. It may, however, be easily observed that any basic feasible solution of an assignment problem contains ($2n - 1$) variables, of which ($n - 1$) variables are zero.

ASSIGNMENT MODEL AS LPP

Let x_{ij} be a variable defined by

$$x_{ij} = \begin{cases} 0 \text{ if the } i\text{th job is not assigned to } j\text{th machine} \\ 1 \text{ if the } i\text{th job is assigned to } j\text{th machine.} \end{cases}$$

Then clearly, since only one job is to be assigned to each machine, we have,

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1,2,\ldots n \quad \text{and} \quad \sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1,2,\ldots,n.$$

Also, the total assignment cost is given by

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Thus, the assignment problem takes the following mathematical form:

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = 1 \quad ; i = 1,2,\ldots,n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad ; j = 1,2,\ldots,n$$

with $x_{ij} = 0$ or 1.

SOLVING ASSIGNMENT PROBLEMS

An assignment problem can be solved by the following four methods:

i. Enumeration Method
ii. Simplex Method
iii. Transportation Method
iv. Hungarian Method.

Enumeration Method

In this method, a list of all possible assignments among the given resources (men, machines, etc.) and activities (jobs, sales areas, etc.,) is prepared. Then an assignment involving the minimum cost (or maximum profit), time or distance is selected. If two or more assignments have the same minimum cost (or maximum profit), time or distance, the problem has multiple optimal solutions.

In general, if an assignment problem involves *n* workers/jobs, then there are in total *n*! possible assignments.

For example, for an $n = 5$ workers/jobs problem, we have to evaluate a total of 5! or 120 assignments. However, when *n* is large, this method is unsuitable for manual calculations. Hence, this method is suitable only for small *n*.

Simplex Method

Since each assignment problem can be formulated as 0 or 1 integer LPP, such a problem can also be solved by the simplex method. In the general mathematical formulation of the assignment problem, there are $n \times n$ decision variables, and $n + n$ or $2n$ equalities. In particular, for a problem involving 5 workers/jobs, there will be 25 decision variables and 10 equalities, which is difficult to solve manually. Hence, the simplex method is not suitable for solving assignment problems.

Transportation Method

Since an assignment is a special case of the transportation problem, it can also be solved by transportation methods. However, every basic feasible solution of a general assignment problem having a square pay-off matrix of order *n* should have $m + n - 1 = n + n - 1 = 2n - 1$ assignments. But due to the special structure of this problem, any solution cannot have more than *n* assignments. Thus, the assignment problem is inherently degenerate. In order to remove degeneracy, $(n - 1)$ number of dummy allocations (deltas or epsilons) will be required in order to proceed with the transportation method. Thus, the problem of degeneracy at each step makes the transportation method computationally inefficient for solving an assignment problem. Hence, the transportation method is not suitable for solving assignment problems.

Hungarian Method

The Hungarian mathematician D Konig developed a simpler and more efficient method of solving assignment problem which is known as the Hungarian technique or method. In this chapter, all assignment problems are solved by the Hungarian method.

The Hungarian method is based on the following principles:

i. If a constant is added to every element of a row/column of the matrix of an assignment problem the resulting assignment problem has the same optimum solution as original problem and vice-versa.
ii. The solution having zero total cost is considered as optimum solution.

Note

i. The AP is usually of minimization type. If the problem is of maximization (i.e., increasing sales/profit) we use the first principle given above to solve the problem.

ii. For minimization AP the second principle is applicable.

Algorithm (Minimization Case)

Step 1: Check whether the number of rows (columns) is equal to number of columns (rows) in the cost matrix. It not, add a dummy row (column) to make it as a square matrix.

Step 2: Starting with the first row, locate the smallest cost element in each row of the cost matrix. Now, subtract this smallest element from each element in that row. As a result, there shall be atleast one zero in each row of this table.

Step 3: In reduced cost matrix obtained in step 2, consider each column and locate smallest element in it. Subtract the smallest element in each column from every element of that column. As a result, there shall be atleast one zero in each of the rows and columns of the second reduced cost table.

Step 4: Make assignments for the reduced matrix obtained in step 2 and 3 in the following manner:

i. Examine the rows successively, until a row with exactly one zero is found. Make an assignment to this single zero and enrectangle $(\square)$ and cross out $(\otimes)$ all other zeroes appearing in the corresponding column as they will not be used to make any other assignment in that column. Proceed in this manner until all rows have been examined.

ii. Examine the columns successively until a column with exactly one zero is found. Make an assignment to this single zero and enrectangle $(\square)$ and cross $(\otimes)$ all other zeroes appearing in the corresponding row. Proceed in this manner until all columns have been examined.

iii. Repeat steps (i) and (ii) until all zeroes in rows and columns are either marked or crossed out. If the number of assignments (marked) made is equal to number of rows/columns, then it is an optimum solution and there will be exactly one assignment in each row and in each column. Otherwise, there will be some rows/columns without assignment. In such case, we proceed to step 5.

Step 5: Draw the minimum number of horizontal and vertical lines necessary to cover all zeroes in the reduced matrix obtained from step 4, in the following way.

a. Mark (✓) all rows that do not have any assignment.

b. Mark (✓) all columns that have zero in marked rows.

c. Mark (✓) all rows (not already marked) that have assignment in marked columns.

d. Repeat steps (a) to (c) until no more rows or columns can be marked.

e. Draw straight lines through all unmarked rows and marked columns.

It may be noted out here that minimum number of lines are to be drawn to cover all zeroes.

Step 6: If the number of lines drawn is equal to '*n*' i.e., equal to number of rows or columns then it is an optimum solution. Otherwise, go to step 7.

Step 7: Select the smallest element among all uncovered elements.

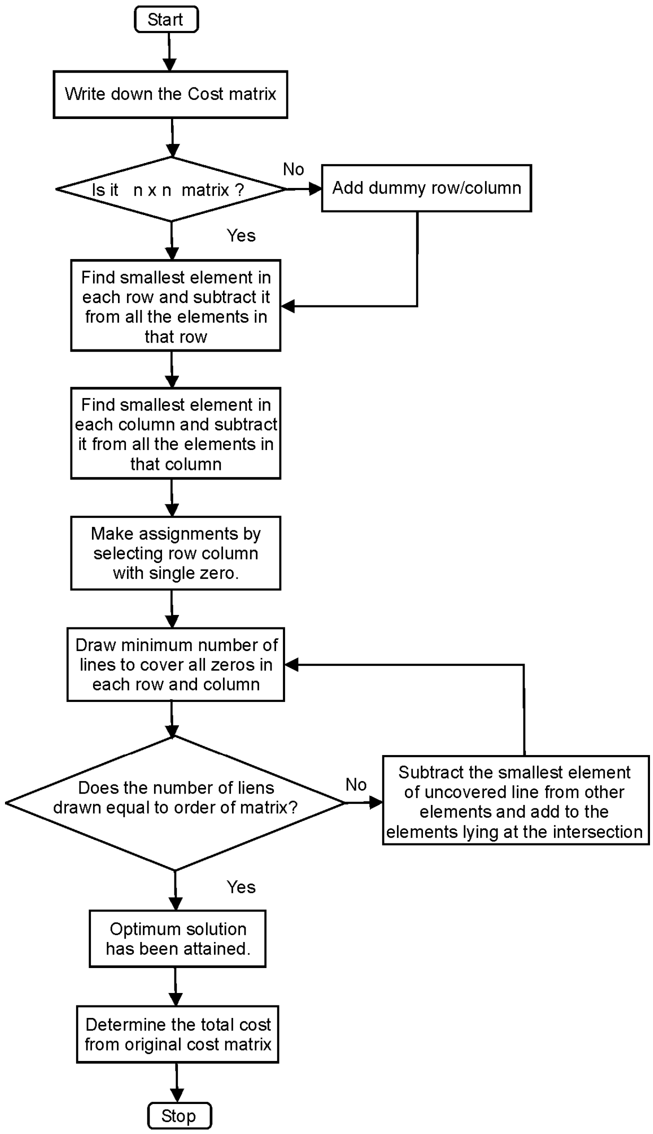
- Subtract this element from all uncovered elements (through which lines are passing).
- Add this element to the element which lies at the intersection $(\dagger)$ of two lines.
- The other elements remain as usual.

Then, we obtain another new reduced matrix for new assignments.

Step 8: Go to step 4 and repeat the procedure until the number of assignments becomes equal to the number of rows or columns. In such case, we shall observe that every row and column have an assignment. Then the current solution is optimal.

Note: In case of maximization problems, after step 1, select the maximum element in the table. Subtract each element of the cost matrix from this maximum element. Follow the same procedure as stated above from step 2 for optimum solution.

Figure: Flow Chart



Example 1

A company is faced with the problem of assigning five jobs to five machines. Each job must be done on only one machine. The cost (in Rs.) of processing each job on each machine is given below.

Jobs ↓	Machines → M_1	M_2	M_3	M_4	M_5
J_1	7	5	9	8	11
J_2	9	12	7	11	10
J_3	8	5	4	6	9
J_4	7	3	6	9	5
J_5	4	6	7	5	11

The problem is to determine the assignment of jobs to machines so that it will result in minimum cost.

Solution

Step 1: Here the number of jobs = number of machines i.e., the cost matrix is a square matrix. So, proceed to step 2.

Step 2: Select the minimum element in each row and subtract this element from every element in that row. Minimum element is 5 in first row, 7 in second row, 4 in third row, 3 in fourth row and 4 in fifth row.

The resultant reduced matrix is shown in Table 1.

Table 1

Jobs ↓	Machines → M_1	M_2	M_3	M_4	M_5
J_1	2	0	4	3	6
J_2	2	5	0	4	3
J_3	4	1	0	2	5
J_4	4	0	3	6	2
J_5	0	2	3	1	7

Step 3: Next, select minimum element in each column and subtract this element from every element in that column. The columns having zero elements will not change (here, column 1, column 2 and column 3). Column 4 and column 5 have minimum element 1 and 2 respectively.

The resultant reduced matrix is shown in Table 2.

Table 2

Jobs ↓	Machines → M_1	M_2	M_3	M_4	M_5
J_1	2	0	4	2	4
J_2	2	5	0	3	1
J_3	4	1	0	1	3
J_4	4	0	3	5	0
J_5	0	2	3	0	5

Step 4: Now, we attempt to make a complete set of assignments using only a single zero element in each row or column. Since row J_1 contains only single zero, an assignment is made in the cell (J_1, M_2) and zeros appearing in the corresponding column M_2 is crossed out. Similarly, we go to next row and find single zero appearing in the 2nd row and assignment is made at (J_1, M_2) and zeroes appearing in the corresponding column M_3 are crossed out.

Now row J_4 has single zero and assignment is made in cell (J_4, M_5). Since there are two zeroes in row J_5, we can not make assignment in that row J_5. Looking columnwise, we find that column M_1 has only single zero and therefore we make an assignment in cell (J_5, M_1) i.e., (C_{51}) and cross-out the zeroes appearing the corresponding row J_5. The assignments so made are shown in Table 3.

Table 3

Jobs ↓	M_1	M_2	M_3	M_4	M_5
J_1	2	$\boxed{0}$	4	2	4
J_2	2	5	$\boxed{0}$	3	1
J_3	4	1	$\cancel{0}$	1	3
J_4	4	$\cancel{0}$	3	5	$\boxed{0}$
J_5	$\boxed{0}$	2	3	$\cancel{0}$	5

Thus, it is possible to make four of the five necessary assignments using the zero element position. So, optimum solution is not reached. Hence, we go to step 5.

Step 5: Draw minimum number of lines, horizontal and vertical, to cover all possible zeroes. Usually all of the zeroes can be obtained by inspection. However, we shall use the method given earlier in explaining the various steps. The various steps in drawing minimum number of lines are:

i. Mark (✓) the row which has no assignment i.e., J_3.

ii. Mark (✓) the column which has zero in marked row i.e., column M_3.

iii. Mark (✓) the row that has assignment in marked column, row J_2.

Drawing the lines through all unmarked rows and marked columns, we get 4(< 5) lines. Hence, the solution is not optimum.

Table 4

Jobs ↓	M_1	M_2	M_3	M_4	M_5	
J_1	2	$\boxed{0}$	4	2	4	
J_2	2	5	$\boxed{0}$	3	1	✓ (3)
J_3	4	1	$\cancel{0}$	1	3	✓ (1)
J_4	4	$\cancel{0}$	3	5	$\boxed{0}$	
J_5	$\boxed{0}$	2	3	$\cancel{0}$	5	
			✓(2)			

Note: The order of marking is indicated by numbers.

Step 6: We examine the elements not covered by these lines and select the smallest element, i.e., 1 is the smallest element not covered by these lines. Subtract this smallest element from all the uncovered elements and add it to the element lying at the intersection of the two lines. The reduced matrix so obtained is shown in Table 5.

Table 5

Jobs ↓ / Machines →	M_1	M_2	M_3	M_4	M_5
J_1	2	0	5	2	4
J_2	1	4	0	2	0
J_3	3	0	0	0	2
J_4	4	0	4	5	0
J_5	0	2	4	0	5

Now, we make fresh assignments. Proceeding in the usual way, the set of assignments made is shown in Table 6.

Table 6

Jobs ↓ / Machines →	M_1	M_2	M_3	M_4	M_5
J_1	2	$\boxed{0}$	5	2	4
J_2	1	4	$\boxed{0}$	2	$\cancel{0}$
J_3	3	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
J_4	4	$\cancel{0}$	4	5	$\boxed{0}$
J_5	$\boxed{0}$	2	4	$\cancel{0}$	5

The number of assignments is 5. There is only one row assignment in each row and each column. Hence, the optimum solution is reached.

Conclusion

The optimum assignment is given below:

Assign Job	To Machine	Cost (Rs.)
J_1	M_2	5
J_2	M_3	7
J_3	M_4	6
J_4	M_5	5
J_5	M_1	4
Minimum Total Cost = Rs.27		

Example 2

A Company is producing a single product and is selling it through five agencies situated in different cities. All of a sudden, there is a demand for the product in another five cities not having any agency of the company. The company is faced with the problem of deciding on how to assign the existing agencies to dispatch the products to needy cities in such a way that the total traveling distance is minimized. The distance between the surplus and deficit cities (in K.M.) is given below:

Surplus Cities ↓	Deficit Cities →				
	P	Q	R	S	T
A	10	5	9	18	11
B	13	19	6	12	14
C	3	2	4	4	5
D	8	9	12	17	15
E	11	6	14	19	10

Solution

Step 1: Here, number of rows = number of columns in the cost matrix. Hence, proceed to step 2.

Step 2: Subtract the minimum element of each row from all elements of that row.

The reduced cost matrix is given below:

Table 7

Surplus Cities ↓	Deficit Cities →				
	P	Q	R	S	T
A	5	0	4	13	6
B	7	13	0	6	8
C	1	0	2	2	3
D	9	0	3	8	6
E	5	0	8	13	4

Step 3: Subtract the minimum element of each column for all elements of that column. The reduced table is given below:

Table 8

Surplus Cities ↓	Deficit Cities →				
	P	Q	R	S	T
A	4	0	4	11	3
B	6	13	0	4	5
C	0	0	2	0	0
D	8	0	3	6	3
E	4	0	8	11	1

Step 4: Making assignments: Rows and columns simultaneously. Taking first row containing single zero and making assignments at AQ (first row and second column) and crossing out all zeroes in column Q. Similarly, making assignment in second row at BR.

Table 9

Surplus Cities ↓	Deficit Cities →				
	P	Q	R	S	T
A	4	$\boxed{0}$	4	11	3
B	6	13	$\boxed{0}$	4	5
C	$\boxed{0}$	$\cancel{0}$	2	$\cancel{0}$	$\cancel{0}$
D	8	$\cancel{0}$	3	6	3
E	4	$\cancel{0}$	8	11	1

Step 5: Only three assignments could be possible. Now drawing minimum number of lines covering all possible zeroes. Following the procedure of drawing lines through unmarked rows and marked columns, we get,

Table 10

Surplus ↓ Cities	Deficit Cities →					
	P	Q	R	S	T	
A	4	$\boxed{0}$	4	11	3	✓ **(4)**
B	6	13	$\boxed{0}$	4	5	
C	$\boxed{0}$	$\cancel{0}$	2	$\cancel{0}$	$\cancel{0}$	
D	8	$\cancel{0}$	3	6	3	✓**(1)**
E	4	$\cancel{0}$	8	11	1	✓**(2)**
		✓**(3)**				

Step 6: Modify the reduced matrix by subtracting the smallest element 1 from all the elements not covered by lines and adding the same at the intersection of the two lines. We get the following matrix:

Table 11

Surplus Cities ↓	Deficit Cities →				
	P	Q	R	S	T
A	3	0	3	10	2
B	6	14	0	4	5
C	0	1	2	0	0
D	7	0	2	5	2
E	3	0	7	10	0

Step 7: Making fresh assignments (in rows and column simultaneously) following the same procedure, we get,

Table 12

Surplus Cities ↓	Deficit Cities →				
	P	Q	R	S	T
A	3	$\boxed{0}$	3	10	2
B	6	14	$\boxed{0}$	4	5
C	$\boxed{0}$	1	2	$\cancel{0}$	$\cancel{0}$
D	7	$\cancel{0}$	2	5	2
E	3	$\cancel{0}$	7	10	$\boxed{0}$

Only four assignments could be made. So, still the optimum solution is not reached. We draw the minimum number of lines.

Table 13

Surplus Cities ↓	Deficit Cities →					
	P	Q	R	S	T	
A	3	$\boxed{0}$	3	10	2	✓**(3)**
B	6	14	$\boxed{0}$	4	5	
C	$\boxed{0}$	1	2	$\cancel{0}$	$\cancel{0}$	
D	7	$\cancel{0}$	2	5	2✓	✓**(1)**
E	3	$\cancel{0}$	7	10	$\boxed{0}$	
		✓ **(2)**				

Modify further the reduced matrix by subtracting the smallest element 2 from all the elements not covered by the lines and adding the same at the intersection of the two lines, we get,

Table 14

Surplus Cities ↓	Deficit Cities →				
	P	Q	R	S	T
A	1	0	1	8	0
B	6	16	0	4	5
C	0	3	2	0	0
D	5	0	0	3	0
E	3	2	7	10	0

Making fresh assignments, we get,

Table 15

Surplus Cities ↓	Deficit Cities →				
	P	Q	R	S	T
A	1	$\boxed{0}$	1	8	$\cancel{0}$
B	6	16	$\boxed{0}$	4	5
C	$\boxed{0}$	3	2	$\cancel{0}$	$\cancel{0}$
D	5	$\cancel{0}$	$\cancel{0}$	3	$\cancel{0}$
E	3	2	7	10	$\boxed{0}$

Still only 4 assignments could be possible. Following the same procedure of marking the rows and columns and drawing lines through unmarked rows and marked columns we get,

Table 16

Surplus Cities↓	Deficit Cities →					
	P	Q	R	S	T	
A	**1**	$\boxed{0}$	1	8	$\cancel{0}$	✓**(4)**
B	6	16	$\boxed{0}$	4	5	✓**(5)**
C	$\boxed{0}$	3	2	$\cancel{0}$	$\cancel{0}$	
D	5	$\cancel{0}$	$\cancel{0}$	3	$\cancel{0}$	✓**(1)**
E	3	2	7	10	$\boxed{0}$	✓**(7)**
		✓	✓		✓	
		(2)	**(3)**		**(6)**	

Modify again the reduced matrix by subtracting the smallest element '1' from all elements not covered by lines and adding the same at the intersection of the lines. We get,

Table 17

Surplus Cities ↓	Deficit Cities →				
	P	Q	R	S	T
A	0	0	1	7	0
B	5	16	0	3	5
C	0	4	3	0	1
D	4	0	0	2	0
E	2	2	7	9	0

Making the fresh assignments, we get,

Table 18

Surplus Cities ↓	Deficit Cities →				
	P	Q	R	S	T
A	$\boxed{0}$	$\cancel{0}$	1	7	$\cancel{0}$
B	5	16	$\boxed{0}$	3	5
C	$\cancel{0}$	4	3	$\boxed{0}$	1
D	4	$\boxed{0}$	$\cancel{0}$	2	$\cancel{0}$
E	2	2	7	9	$\boxed{0}$

Since the number of assignments are five, and in each row and each column there is only one assignment, the optimum assignment schedule has reached.

Conclusion

The optimum assignment is given below:

Assign Job	To Machine	Cost (Rs.)
A	P	10
B	Q	6
C	R	4
D	S	9
E	T	10
Minimum Distance = 39 K.M.		

MAXIMIZATION CASE IN ASSIGNMENT PROBLEM

In some cases, the pay-off elements of the assignment problem may represent revenue or profits instead of costs, so that the objective will be to maximize total revenue or profit. The Hungarian method explained earlier can also be used for such maximization case. The problem of maximization can be converted into minimization case by selecting the largest element among all elements of the profit matrix and then subtracting each element of the matrix from it, including itself. We can proceed for optimum solution using the Hungarian algorithm and obtain the maximum by adding the original values of these cells to which the assignments have been made.

Example 3

A director in a management institute has the problem of assigning courses to teachers with a view to maximize educational quality in his institute. He has available with him one professor, two associate professors, and one Teaching Assistance (TA). Four courses must be cleared and, after appropriate evaluation, has arrived at the following relative ratings (100 = best rating) regarding the ability of each instructor to teach each of the four courses.

Professor	Courses			
	1	2	3	4
1	60	40	60	70
2	20	60	50	70
3	20	30	40	60
TA	30	10	20	40

How should he assign his staff to the courses to realize his objective?

Solution

This is the maximization problem. So, we convert it into minimization problem by subtracting each element of effectiveness matrix from the maximum element '70'.

Professor	Courses 1	2	3	4
1	10	30	10	0
2	50	10	20	0
3	50	40	30	10
TA	40	60	40	30

Here, the number of professors = number of courses = number of rows/columns. Therefore, proceed to row reduction.

Row Reduction

Professor	Courses 1	2	3	4
1	10	30	10	0
2	50	10	20	0
3	40	30	20	0
TA	10	30	10	0

Column Reduction

Professor	Courses 1	2	3	4
1	0	20	0	0
2	40	0	10	0
3	30	20	10	0
TA	0	20	0	0

Making Assignments

Professor	Courses 1	2	3	4
1	$\boxed{0}$	20	$\cancel{0}$	$\cancel{0}$
2	40	$\boxed{0}$	10	$\cancel{0}$
3	30	20	10	$\boxed{0}$
TA	$\cancel{0}$	20	$\boxed{0}$	$\cancel{0}$

Since, the number of assignments = number of rows/columns, optimality has been attained.

Hence, the director should assign his staff in the following manner in order to realize the objective i.e., maximizing the education quality.

Prof. 1 $\rightarrow$ Course 1, Prof. 2 $\rightarrow$ Course 2, Prof. 3 $\rightarrow$ Course 4,

TA $\rightarrow$ Course 3 with maximum rating = 200.

TRAVELING SALESMAN PROBLEM

Traveling salesman problem is very similar to the assignment problem except that in the former there is an additional restriction that a salesman who starts from his home city, visits each city only once and returns to his home city.

Example 4

Solve the following traveling salesman problem so as to minimize the cost per cycle.

From \ To	A	B	C	D	E
A	–	3	6	2	3
B	3	–	5	2	3
C	6	5	–	6	4
D	2	2	6	–	6
E	3	3	4	6	–

Solution

Here, the number of salesman = number of routes = number of rows/columns. Therefore, proceed to row reduction.

Row Reduction

Subtract the smallest element of each row from other elements of the same row and assign ∞ in the prohibited cell.

From \ To	A	B	C	D	E
A	∞	1	4	0	1
B	1	∞	3	0	1
C	2	1	∞	2	2
D	0	0	4	∞	4
E	0	0	1	3	∞

Column Reduction

From \ To	A	B	C	D	E
A	∞	1	3	0	1
B	1	∞	2	0	1
C	2	1	∞	2	0
D	0	0	3	∞	4
E	0	0	0	3	∞

Make zero assignments and draw the minimum number of lines to cover all the zeroes after marking rows and columns and drawing lines through unmarked rows and marked columns.

From \ To	A	B	C	D	E	
A	∞	1	3	$\boxed{0}$	1	✓(3)
B	1	∞	2	$\cancel{0}$	1	✓(1)
C	2	1	∞	2	$\boxed{0}$	
D	$\boxed{0}$	$\cancel{0}$	3	∞	4	
E	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	3	∞	
				✓(2)		

Modify the table by subtracting the smallest element '1' from all the elements not covered by lines and adding the same at the intersection of lines. Further, make assignments.

To / From	A	B	C	D	E
A	∞	$\boxed{0}$	2	$\cancel{0}$	$\cancel{0}$
B	$\cancel{0}$	∞	1	$\boxed{0}$	$\cancel{0}$
C	2	1	∞	3	$\boxed{0}$
D	$\boxed{0}$	$\cancel{0}$	3	∞	4
E	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	4	∞

The assignment is $A \to B,\ B \to D,\ D \to A$ and is not feasible (as he is visiting home city A without visiting C and E).

Since this assignment schedule does not provide us the solution of travelling salesman problem, we try to find the next best solution satisfying the extra condition also.

Alternatively, make assignment at cell BC instead of BD, selecting 1 for assignment (i.e., the next higher value 1 in the matrix).

To / From	A	B	C	D	E
A	∞	$\cancel{0}$	2	$\boxed{0}$	$\cancel{0}$
B	$\cancel{0}$	∞	$\boxed{1}$	$\cancel{0}$	$\cancel{0}$
C	2	1	∞	3	$\boxed{0}$
D	$\cancel{0}$	$\boxed{0}$	3	∞	4
E	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	4	∞

The schedule is $A \to D \to B \to C \to E \to A$.

OR

To / From	A	B	C	D	E
A	∞	$\cancel{0}$	2	$\cancel{0}$	$\boxed{0}$
B	$\cancel{0}$	∞	$\cancel{1}$	$\boxed{0}$	$\cancel{0}$
C	2	$\boxed{1}$	∞	3	$\cancel{0}$
D	$\boxed{0}$	$\cancel{0}$	3	∞	4
E	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	4	∞

$A \to E \to C \to B \to D \to A$

Minimum cost is Rs.16.

Conclusion

The optimum schedule is $A \to E \to C \to B \to D \to A$, since all the routes are covered. Also there is another assignment route $A \to D \to B \to C \to E \to A$. Both the routes give rise to the same minimum cost of Rs.16.

Example 5

Solve the traveling salesman problem given by the following data:

$C_{12} = 20,\ C_{13} = 4,\ C_{14} = 10,\ C_{23} = 5,\ C_{34} = 6,\ C_{25} = 10,\ C_{35} = 6,\ C_{45} = 20$

where, $C_{ij} = C_{ji}$ and there is no route between cities i and j if a value of C_{ij} is not known.

Solution

We take $C_{ij} = \infty$ for $i = j$ as there is no route when $i = j$.

The given problem can be expressed in the form of an assignment problem.

From \ To	1	2	3	4	5
1	∞	20	4	10	∞
2	20	∞	5	∞	10
3	4	5	∞	6	6
4	10	∞	6	∞	20
5	∞	10	6	20	∞

Row Reduction

From \ To	1	2	3	4	5
1	∞	16	0	6	∞
2	15	∞	0	∞	5
3	0	1	∞	2	2
4	4	∞	0	∞	14
5	∞	4	0	14	∞

Column Reduction

From \ To	1	2	3	4	5
1	∞	15	0	4	∞
2	15	∞	0	∞	3
3	10	0	∞	0	0
4	4	∞	0	∞	12
5	∞	3	0	12	∞

Making Assignment and Drawing Lines

From \ To	1	2	3	4	5	
1	∞	15	$\boxed{0}$	4	∞	✓(5)
2	15	∞	$\cancel{0}$	∞	3	✓(1)
3	$\boxed{0}$	$\cancel{0}$	∞	$\cancel{0}$	$\cancel{0}$	
4	4	∞	$\cancel{0}$	∞	12	✓(2)
5	∞	3	$\cancel{0}$	12	∞	✓(3)
			✓(3)			

Subtract the smallest uncovered element from other uncovered elements and add it to the element at the intersection of lines.

To / From	1	2	3	4	5	
1	∞	12	$\boxed{0}$	1	∞	✓(4)
2	12	∞	$\cancel{0}$	∞	0	✓(1)
3	$\boxed{0}$	$\cancel{0}$	∞	$\cancel{0}$	$\cancel{0}$	
4	1	∞	$\cancel{0}$	∞	9	✓(2)
5	∞	$\boxed{0}$	$\cancel{0}$	9	∞	

✓(3)

Subtracting smallest uncovered element from other uncovered elements and adding it to the element at the intersection of lines, we get

To / From	1	2	3	4	5
1	∞	11	$\boxed{0}$	$\cancel{0}$	∞
2	12	∞	1	∞	$\boxed{0}$
3	$\cancel{0}$	$\cancel{0}$	∞	$\boxed{0}$	$\cancel{0}$
4	$\boxed{0}$	∞	$\cancel{0}$	∞	9
5	∞	$\boxed{0}$	1	9	∞

The optimum assignment is $1 \rightarrow 3 \rightarrow 4 \rightarrow 1$. Hence, not feasible.

Selecting 1 as the next minimum number to zero, the next best solution which satisfy the extra restriction is as follows:

To / From	1	2	3	4	5
1	∞	11	$\cancel{0}$	$\boxed{0}$	∞
2	12	∞	$\boxed{1}$	∞	$\cancel{0}$
3	$\cancel{0}$	$\cancel{0}$	∞	$\cancel{0}$	$\cancel{0}$
4	$\boxed{0}$	∞	$\cancel{0}$	∞	$\boxed{8}$
5	∞	$\boxed{8}$	1	9	∞

The revised optimum assignment is $1 \rightarrow 4 \rightarrow 1$, which is not feasible. Therefore,

Select '8' as next minimum number to 1 and zero. Now, make assignment at '8' , '1'and' 0's.

To / From	1	2	3	4	5
1	∞	11	$\cancel{0}$	$\boxed{0}$	∞
2	12	∞	$\boxed{1}$	∞	$\cancel{0}$
3	$\boxed{0}$	$\cancel{0}$	∞	$\cancel{0}$	$\cancel{0}$
4	$\cancel{0}$	∞	$\cancel{0}$	∞	$\boxed{8}$
5	∞	$\boxed{0}$	$\cancel{0}$	9	∞

Conclusion

The optimum sequence for traveling is $1 \to 4 \to 5 \to 2 \to 3 \to 1$.

The total set up cost according to the above assignment schedule is

10 + 5 + 4 + 20 + 10 = 49.

UNBALANCED ASSIGNMENT PROBLEM

Whenever the cost matrix of assignment problem is not a square matrix i.e., number of rows is not equal to number of columns, the assignment problem is called unbalanced assignment problem. In such cases, dummy rows/columns are added to the matrix to make it a square matrix. Then, we apply the Hungarian method to this resultant (square matrix) assignment problem. For example, if five workers are to be assigned to six machines, a dummy worker is simply added to transform the assignment problem into (6×6) matrix. The cost or time associated with this dummy row or column is assigned zero elements in the matrix.

Example 6

A company is faced with the problem of assigning six different machines to five different jobs. The costs are estimated and given below. Assign the jobs in order to minimize the cost. Is there any machine idle?

Machines ↓	Jobs →				
	J_1	J_2	J_3	J_4	J_5
M_1	6	2	5	2	6
M_2	2	5	8	7	7
M_3	7	8	6	9	8
M_4	6	2	3	4	5
M_5	9	3	8	9	7
M_6	4	7	4	6	8

Solution

Since the number of machines is not equal to number of jobs, a dummy job J_6 is created. The cost associated with any machine for this dummy job is zero. Thus, the square matrix is as shown below:

Machines ↓	Jobs →					
	J_1	J_2	J_3	J_4	J_5	J_6
M_1	6	2	5	2	6	0
M_2	2	5	8	7	7	0
M_3	7	8	6	9	8	0
M_4	6	2	3	4	5	0
M_5	9	3	8	9	7	0
M_6	4	7	4	6	8	0

Now, every row has zero element, there is no need of row reduction. Hence, we straight away proceed to column reduction step.

Column Reduction

Machines ↓	Jobs →					
	J_1	J_2	J_3	J_4	J_5	J_6
M_1	4	0	0	0	1	0
M_2	0	3	5	5	2	0
M_3	5	6	3	7	3	0
M_4	4	0	2	0	0	0
M_5	7	1	5	7	2	0
M_6	2	5	1	4	3	0

Making assignments, we observe that only four assignments are possible. To create more zeroes, we draw minimum number of lines after marking rows and columns.

Machines ↓	Jobs →						
	J_1	J_2	J_3	J_4	J_5	J_6	
M_1	4	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	1	$\cancel{0}$	
M_2	$\boxed{0}$	3	5	5	2	$\cancel{0}$	
M_3	5	6	3	7	3	$\cancel{0}$	✓**(1)**
M_4	4	$\boxed{0}$	2	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	
M_5	7	1	5	7	2	$\boxed{0}$	✓**(4)**
M_6	2	5	1	4	3	$\cancel{0}$	✓**(2)**
						✓**(3)**	

Subtracting the smallest uncovered element i.e., '1' from all uncovered elements and adding it to elements at intersection of lines, we get the reduced matrix. Making assignments at single zero element in row/column, we get six assignments as shown below.

Machines ↓	Jobs →					
	J_1	J_2	J_3	J_4	J_5	J_6
M_1	4	$\cancel{0}$	2	$\boxed{0}$	1	1
M_2	$\boxed{0}$	3	5	5	2	1
M_3	4	5	2	6	2	$\boxed{0}$
M_4	4	$\cancel{0}$	$\cancel{0}$	2	$\boxed{0}$	1
M_5	6	$\boxed{0}$	4	6	1	$\cancel{0}$
M_6	1	4	$\boxed{0}$	3	2	$\cancel{0}$

Conclusion

The optimum assignments made are

$M_1 \rightarrow J_4$; $M_2 \rightarrow J_1$; $M_3 \rightarrow J_6$ (Dummy);

$M_4 \rightarrow J_5$; $M_5 \rightarrow J_2$; $M_6 \rightarrow J_3$

Minimum cost = Rs.16

From the above, it is clear that machine 3 is idle and will not be assigned any job.

Example 7

A production manager wants to assign one of the five new methods to each of the four operations. The following table summarizes the weekly output in units:

Operator ↓   Weekly Output →

	M_1	M_2	M_3	M_4	M_5
A	4	6	11	16	9
B	5	8	16	19	9
C	9	13	21	21	13
D	6	6	9	11	7

Cost per unit is Rs.20, selling price per unit is Rs.30. Find the maximum profit per month.

Solution

A dummy operator is introduced to make it balanced matrix.

Operator ↓   Weekly Output →

	M_1	M_2	M_3	M_4	M_5
A	4	6	11	16	9
B	5	8	16	19	9
C	9	13	**21**	21	13
D	6	6	9	11	7
Dummy	0	0	0	0	0

Loss (cost) matrix is prepared by subtracting all elements from the largest element 21 since the problem is of maximization of output.

Operator ↓   Weekly Output →

	M_1	M_2	M_3	M_4	M_5
A	17	15	10	5	12
B	16	13	5	2	12
C	12	8	0	0	8
D	15	15	12	10	14
Dummy	21	21	21	21	21

Row Reduction

Operator ↓ Weekly Output →

	M_1	M_2	M_3	M_4	M_5
A	12	10	5	0	7
B	14	11	3	0	10
C	12	8	0	0	8
D	5	5	2	0	4
Dummy	0	0	0	0	0

Column reduction is not required since each column has zero elements.

Making Assignments

Operator ↓ Weekly Output →

	M_1	M_2	M_3	M_4	M_5
A	12	10	5	$\boxed{0}$	7
B	14	11	3	$\cancel{0}$	10
C	12	8	$\boxed{0}$	$\cancel{0}$	8
D	5	5	2	$\cancel{0}$	4
Dummy	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

Since the number of lines are 3 as against required 5, we proceed to first reduction.

Operator ↓ Weekly Output →

	M_1	M_2	M_3	M_4	M_5
A	8	6	5	$\boxed{0}$	3
B	10	7	3	$\cancel{0}$	6
C	8	4	$\boxed{0}$	$\cancel{0}$	4
D	1	1	2	$\cancel{0}$	$\boxed{0}$
Dummy	$\boxed{0}$	$\cancel{0}$	4	4	$\cancel{0}$

First Reduction

Operator ↓ Weekly Output →

	M_1	M_2	M_3	M_4	M_5
A	5	3	2	0	0
B	7	4	0	0	3
C	8	4	0	0	3
D	1	1	2	3	0
Dummy	$\boxed{0}$	$\cancel{0}$	4	7	$\cancel{0}$

Since the number of lines are 4 against required 5 we proceed to second reduction.

Operator $\downarrow$ Weekly Output $\rightarrow$

$$\begin{array}{c|ccccc|}
 & M_1 & M_2 & M_3 & M_4 & M_5 \\
\hline
A & 4 & 2 & 2 & \cancel{0} & \boxed{0} \\
B & 6 & 3 & \cancel{0} & \boxed{0} & 3 \\
C & 7 & 3 & \boxed{0} & 3 & 4 \\
D & \boxed{0} & \cancel{0} & 2 & 3 & 4 \\
\text{Dummy} & \cancel{0} & \boxed{0} & 5 & 8 & 1 \\
\hline
\end{array}$$

Conclusion

The Optimal assignment is

$$A \rightarrow M_5;\ B \rightarrow M_4;\ C \rightarrow M_3;\ D \rightarrow M_1;\ Dummy \rightarrow M_2$$

Minimum Output = 55 per week.

Production per month $= 55 \times 4 = 220$.

Profit per unit = 30 – 20 =10.

Profit per month = 220×10 = Rs.2,200.

CREW ASSIGNMENT PROBLEM

The method of solution discussed in this chapter can be utilized to plan the assignment of crew members in different locations by a transport company. The technique is illustrated with the help of the following examples:

Example 8

Prachi Air lines that operates seven days a week has a time table shown below. Crews must have a minimum layover of 6 hours between flights. Obtain the pairing of flights that minimizes layover time away home. For any given pairing the crew will be based at the city that results in the smaller layover.

Flight No.	Chennai Depart.	Mumbai Arrival	Flight No.	Mumbai Depart.	Chennai Arrival
1	7.00A.M.	9.00A.M.	101	9.00A.M.	11.00A.M.
2	9.00A.M.	11.00A.M.	102	10.00A.M.	12Noon
3	1.30P.M.	3.30P.M.	103	3.30P.M.	5.30P.M.
4	7.30P.M.	9.30P.M.	104	8.00P.M.	10.00P.M.

For each pair, also mention the town where the crew should be based.

Solution

The following table gives the layover time of crew based at Chennai:

Table – I

	101	102	103	104
1	24	25	6.5	11
2	22	23	28.5	28.5
3	17.5	8.5	24	28.5
4	11.5	12.5	18	22.5

Calculation of time is explained as below (Row wise computations):

Route $1 \to 101,\ 1 \to 102,\ 1 \to 103,\ 1 \to 104$

9.00 AM → 9.00 AM (24 hrs.)
→ 10.00 AM (25 hrs.)
→ 3.30 PM (6.5 hrs.)
→ 8. 00 PM (11 hrs.)

Remark

For 3. 30 PM and 8.00PM, length of the time is taken on the same day.

Route $2 \to 101,\ 2 \to 102,\ 2 \to 103,\ 2 \to 104$

11.00 AM → 9.00 AM (22 hrs.)
→ 10.00 AM (23 hrs.)
→ 3.30 PM (28.5 hrs.)
→ 8. 00 PM (28.5 hrs.)

Route $3 \to 101,\ 3 \to 102,\ 3 \to 103, 32 \to 104$

3.30 PM → 9.00 AM (17.5 hrs.)
→ 10.00 AM (18.5 hrs.)
→ 3.30 PM (24 hrs.)
→ 8. 00 PM (28.5 hrs.)

Route $4 \to 101,\ 4 \to 102,\ 4 \to 103,\ 4 \to 104$

9.30 PM → 9.00 AM (11.5 hrs.)
→ 10.00 AM (12.5 hrs.)
→ 3.30 PM (18 hrs.)
→ 8. 00 PM (22 .5 hrs.)

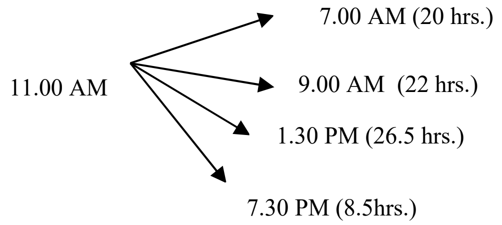
The following table gives the layover time of crew based at Mumbai:

Table – 20

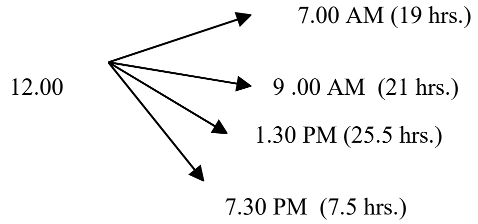
	101	102	103	104
1	20	19	13.5	9
2	22	21	15.5	11
3	26.5	25.5	20	15.5
4	8.5	7.5	26	21.5

Calculation of time is similar to the above but in column direction:

Route $101 \to 1,\ 102 \to 2,\ 103 \to 3,\ 104 \to 4$



Route $102 \to 1,\ 102 \to 2,\ 102 \to 3,\ 102 \to 4$



Similarly, it is calculated for the third and fourth rows.

Minimum Layover Table

	101	102	103	104
1	20^M	19^M	6.5^C	9^M
2	$22^{C/M}$	21^M	15.5^C	9^C
3	17.5^M	18.5^C	20^M	15.5^M
4	8.5^M	7.5^M	18^C	21.5^M

Minimum layover time is obtained from the tables I and II.

For the route $1 \to 101$ i.e., for the cell (1,1), min (24, 20) = 20 and this minimum is from Table II i.e., crew based at Mumbai. Hence, it is indicated as 20^M at this cell.

For the route $1 \to 103$ i.e., for the cell (1,3), min (6.5, 13.5) = 6.5 and this minimum is from Table I i.e., the crew based at Chennai. Hence, it is indicated as 6.5^C at this cell.

For the route $2 \to 101$ i.e., for the cell (2,1), min (22, 22) = 22 and this minimum is from both the tables i.e., the crew based at Chennai and the crew based at Mumbai. Hence, it is indicated as $22^{C/M}$ at this cell.

Note: Super script M indicates Mumbai, C indicates Chennai and C/M indicates both Chennai and Mumbai.

Now the Hungarian method is applied to this problem.

Row Reduction

	101	102	103	104
1	13.5	12.5	0	2.5
2	13	12	6.5	0
3	2	3	4.5	0
4	1	0	10.5	14

Column Reduction

	101	102	103	104
1	12.5	12.5	0	2.5
2	12	12	6.5	0
3	1	3	4.5	0
4	0	0	10.5	14

Making Assignments

	101	102	103	104	
1	12.5	12.5	$\boxed{0}$	2.5	
2	12	12	6.5	$\boxed{0}$	**✓(3)**
3	1	3	4.5	$\cancel{0}$	**✓(1)**
4	$\boxed{0}$	$\cancel{0}$	10.5	14	

Subtracting the smallest uncovered element from all uncovered elements and adding it to element at the intersection of lines we get,

	101	102	103	104
1	12.5	12.5	$\boxed{0}$	3.5
2	11	11	5.5	$\boxed{0}$
3	$\boxed{0}$	2	3.5	$\cancel{0}$
4	$\cancel{0}$	$\boxed{0}$	10.5	15

Conclusion

The Optimum schedule is

Flight	Crew base	Layover time
$1 \rightarrow 103$	Chennai	6.5
$2 \rightarrow 104$	Chennai	9.0
$3 \rightarrow 101$	Chennai	17.5
$4 \rightarrow 102$	Mumbai	7.5
		40.5 hours

Example 9

A trip from Madras to Bangalore takes 6 hours by bus. A typical time table of the bus service in both directions is given below:

Departure from Madras	Route Number	Arrival at Bangalore	Arrival at Madras	Route Number	Departure from Bangalore
06.00	a	12.00	11.30	1	05.30

07.30	b	13.30	15.00	2	09.00
11.30	c	17.30	21.00	3	15.00
19.00	d	01.00	00.30	4	18.30
00.30	e	06.30	06.00	5	00.00

The cost of providing this service by the transport company depends upon the time spent by the bus crew (driver and conductor) away from their places in addition to service time. There are five crews. There is a constraint that every crew should be provided with more than 4 hours of rest before the return trip again, and should not wait for more than 24 hours for the return trip. The company has residential facilities for the crew at Madras as well as at Bangalore. Find which crew be assigned to which line of service so as to reduce the waiting time to the minimum.

Solution

As the service time is constant, it does not effect the decision of starting the crew. Now, if the crew resides at Madras, they start from Madras and come back to Madras with minimum stay at Bangalore, then waiting time at Bangalore for different service line connections may be calculated as given below.

Waiting Time (in hours) at Bangalore

Route \ Time	1	2	3	4	5
a	17.5	21	∞	6.5	12
b	16	19.5	∞	5	10.5
c	12	15.5	21.5	∞	6.5
d	4.5	8	14	17.5	23
e	23	∞	8.5	12	17.5

Similarly if the crew is assumed to reside at Bangalore, so that they start from and come back to Bangalore, then the waiting time for different route connections would follow.

Waiting Time (in hours) at Madras

Route \ Time	1	2	3	4	5
a	18.5	15	9	5.5	∞
b	20	16.5	10.5	7	∞
c	∞	20.5	14.5	11	5.5
d	7.5	∞	22	18.5	13
e	13	9.5	∞	∞	18.5

Now since the crew can be asked to reside at either of the places, minimum waiting times from the above operation can be obtained for different route connections by selecting the corresponding lower value out of the above two waiting times, provided the waiting time is greater than 4 hours. The resulting waiting time matrix is as given below:

Route \ Time	1	2	3	4	5
a	17.5^M	15	9	5.5	12^M
b	16^M	16.5	10.5	5^M	10.5^M
c	12^M	15.5^M	14.5	11	5.5

d	4.5^M	8^M	14^M	17.5^M	13
e	13	9.5	8.5^M	12^M	17.5^M

(the layover time marked with (^{M}) denote that the crew is based at Madras, otherwise, the crew based at Bangalore)

We shall the apply Hungarian Method (HAM) to solve this as an assignment problem.

Row Reduction

Deducting the minimum value of each row from all the values of that row, we get,

Time / Route	1	2	3	4	5
a	12	9.5	3.5	0	6.5
b	11	11.5	5.5	0	5.5
c	6.5	10	9	5.5	0
d	0	3.5	9.5	13	8.5
e	4.5	1	0	3.5	9

Column Reduction

Deducting the minimum value of the column from all the values of that column and covering all zeroes with lines, we get,

Time / Route	1	2	3	4	5
a	12*	9.5	**3.5**	0	6.5*
b	11*	11.5	5.5	0*	5.5*
c	6.5*	9*	9	5.5	0
d	0*	2.5	9.5*	13*	8.5
e	4.5	0	0*	3.5*	9*

Since the number of lines covering all the zeroes is 4 which is less than 5, we subtract the smallest uncovered element 3.5 from all the values which are not covered by any line, and add it to the values at the point of intersection. Thus we get,

Time / Route	1	2	3	4	5
a	8.5*	5	0	0	3*
b	7.5*	7	2	0	2*
c	6.5*	9*	9	0	0
d	0*	2.5*	9.5*	16.5*	8.5
e	4.5	0	0*	7*	9*

In the above matrix, all the zeroes are covered by 5 lines as against the required 5. Hence, the optimum assignment can be made as follows:

Time / Route	1	2	3	4	5
a	8.5*	5	$\boxed{0}$	$\cancel{0}$	3*

b	7.5*	7	2	$\boxed{0}$	2*
c	6.5*	9*	9	$\cancel{0}$	$\boxed{0}$
d	$\boxed{0*}$	2.5*	9.5*	16.5*	8.5
e	4.5	$\boxed{0}$	$\cancel{0}$	7*	9*

Conclusion

The optimum assignment of the crew should be

Crew	Based at	Route No.	Waiting Time
1	Madras	$1 \rightarrow d$	4.5 hours
2	Bangalore	$2 \rightarrow e$	9.5 Hours
3	Bangalore	$3 \rightarrow a$	9.0 Hours
4	Madras	$4 \rightarrow b$	5.0 Hours
5	Bangalore	$5 \rightarrow c$	5.0 Hours
			33.0 Hours

SUMMARY

- Assignment problem deals with allocating the various resources or items to various activities on one-to-one basis in such a way that the time or cost involved is minimized and the sale or profit is maximized.
- Assignment problem is a special type of linear programming problem.
- Assignment problem is solved by the Hungarian Technique.
- A problem of maximization can be converted into minimization case by selecting the largest element among all elements of the profit matrix and then subtracting each element in the matrix from the maximum element, including itself.
- In the traveling salesman problem, a salesman who starts from his home city, visit each city only once and returns to his home city.
- An assignment problem is said to be unbalanced, if the effectiveness matrix is not a square matrix.
- In the unbalanced assignment problem, dummy rows/columns are added in the matrix to make it a square matrix.

Exercise

1. What is an assignment problem? Explain.
2. Give the mathematical formulation of an assignment problem.
3. What is an assignment problem and how do you interpret it as a LPP?
4. Explain the difference between a transportation problem and an assignment problem.
5. Give in detail the computational procedure of solving an assignment problem.
6. If in an assignment problem we add a constant to every element of a row of the cost matrix, then prove that an assignment plan which minimizes the total cost for the new matrix, also minimizes the total cost for the original cost matrix.
7. Discuss the Hungarian method of solving an assignment problem.
8. Can there be multiple solutions to an assignment problem? How would you identify the existence of multiple solutions, if any?
9. How would you deal with the assignment problems, where (a) the objective function is to be maximized? (b) some assignments are prohibited?
10. Describe the method of drawing minimum number of lines in the context of assignment problem. Name the method.
11. Explain the method of solving a maximization assignment model.
12. Explain the nature of traveling salesman problem and give its mathematical formulation.
13. Write a short note on Traveling Salesman Problem.
14. A project has four men available for work on four separate jobs. Only one man can work on any one job. The cost of assigning each man to each job is given in the following table. Assign men to jobs such that the total cost of assignment is minimum.

Jobs / Men	1	2	3	4
A	20	25	22	28
B	15	18	23	17
C	19	17	21	24
D	25	23	24	24

15. A tourist car rental firm has one car in each of the five depots D_1, D_2, D_3, D_4, D_5 and a customer in each of the five cities C_1, C_2, C_3, C_4, C_5. The distances in kilometers between the depots and the cities are given in the following matrix. How should cars be assigned to the customers so as to minimize the total distance covered?

Depots	Cities				
	C_1	C_2	C_3	C_4	C_5
D_1	140	115	120	30	35
D_2	110	100	90	30	15
D_3	155	90	135	60	50
D_4	170	140	150	60	60
D_5	180	155	165	90	85

16. Solve the following unbalanced assignment problem of minimizing the total time for performing all the jobs.

Operator	Job				
	1	2	3	4	5
1	6	2	5	2	6
2	2	5	8	7	7
3	7	8	6	9	8
4	6	2	3	4	5
5	9	3	8	9	7
6	4	7	4	6	8

17. The personnel manager of a company wants to assign Mr. X, Mr. Y and Mr. Z to regional offices Delhi, Bombay, Calcutta and Madras. The cost of relocation (in Rupees) of the three officers at the four regional offices are given below:

Officer	Office			
	Delhi	Bombay	Calcutta	Madras
Mr. X	16000	22000	24000	20000
Mr. Y	10000	32000	26000	16000
Mr. Z	10000	20000	46000	30000

How would you assign each officer to an office so that the cost is minimum?

18. A metal shop has five jobs to be done and his five machines to do them. The cost matrix gives the cost of processing each job on any machine. Because of specific job requirement and machine specifications certain jobs cannot be done on certain machines. These have been shown by X in the cost matrix. The assignment of jobs to machines must be done on a one-to-one basis. The objective is to assign the jobs to the machines so as to minimize the total cost within the restrictions mentioned above.

Machines	Jobs				
	1	2	3	4	5
1	80	40	X	70	40
2	X	80	60	40	40
3	70	X	60	80	70
4	70	80	30	50	X
5	40	40	50	X	80

19. Five different machines can do any of the required five jobs with different profits resulting from each assignment as shown below:

Machines	Jobs				
	A	B	C	D	E
1	30	37	40	28	40
2	40	24	27	21	36
3	40	32	33	30	35
4	25	38	40	36	36
5	29	62	41	34	39

Find out the maximum profit possible through optimal assignment.

20. Six contractors have submitted tenders to take up six projects advertised. It is noted that one contractor can be assigned one job as otherwise the time for

completion and the quality of workmanship will be affected. The estimates of cost in thousand rupees given by each of them are indicated below:

Contractor	Project					
	1	2	3	4	5	6
A	41	72	39	52	25	51
B	22	29	49	65	81	50
C	27	39	60	51	32	32
D	45	50	48	52	37	43
E	29	40	39	26	30	33
F	82	40	40	60	51	30

Find out the assignment such that the total cost of completing the six projects is minimum. What is the minimum cost?

21. A freight terminal can accommodate six trucks simultaneously. There is cost of sorting and transferring of loads with parking of each truck on each of the six sports. On a certain day, four trucks are to be simultaneously parked at the terminal. The cost matrix is given below:

Spot	Truck			
	1	2	3	4
7	3	6	2	6
8	7	1	4	4
9	3	8	5	8
10	6	4	3	7
11	5	2	4	3
12	5	7	6	2

Find out the assignment which minimizes the total cost of parking.

22. A sales manager has to assign salesman to four territories. He has four candidates of varying experience and capabilities and assesses the possible profit in suitable units for each salesman in each territory as given below:

Salesman	Territories			
	T_1	T_2	T_3	T_4
S_1	25	27	28	37
S_2	28	34	29	40
S_3	35	24	32	33
S_4	24	32	25	28

Find an assignment that maximizes the profit.

23. A machine operator processes five of items on his machine each week, and must choose a sequence for them. The set-up cost per change depends on the

item presently on the machine and the set-up to be made according to the following table:

From item	To item				
	A	B	C	D	E
A	∞	4	7	3	4
B	4	∞	6	3	4
C	7	6	∞	7	5
D	3	3	7	∞	7
E	4	4	5	7	∞

If he processes each type of item once and only once each week, how should he sequence the items on his machine in order to minimize the total set-up cost?

Key to Exercises

Chapter II

7. $x_1 = 0,\ x_2 = 3.2$, Max $Z = 32.$

8. $x_1 = 4,\ x_2 = 0$, Max $Z = 20,00,000.$

9. $x_1 = 12.5,\ x_2 = 35$, Max $Z = 2150.$

10. Max $Z = 8x_1 + 5x_2$

    Subject to –

    $2x_1 + x_2 \le 500$

    $x_1 \le 150$

    $x_2 \le 250$

    $x_1, x_2 \ge 0.$

    Ans: $x_1 = 125,\ x_2 = 250$, Max $Z = 2250.$

11. Max $Z = 3x_1 + 5x_2$

    Subject to –

    $x_1 + 2x_2 \le 2000$ (time constraint)

    $x_1 + x_2 \le 1500$ (plastic constraint)

    $x_2 \le 600$ (dress constraint)

    $x_1, x_2 \ge 0.$

    Ans: $x_1 = 1000,\ x_2 = 500$, Max $Z = 5500.$

12. Max $Z = 3x_1 + 5x_2 + 4x_3$

    Subject to –

    $2x_1 + 3x_2 \le 8$

    $2x_2 + 5x_3 \le 10$

    $3x_1 + 2x_2 + 4x_3 \le 15$

    $x_1, x_2, x_3 \ge 0.$

    This problem cannot be solved by graphical technique, since it has three variables. The solution by simplex is given as follows:

    $x_1 = 89/41,\ x_2 = 50/41, x_3 = 62/41$, Max $Z = 765/41.$

13. Max $Z = 1850x_1 + 2080x_2 + 1875x_3$

    Subject to –

    $x_1 + x_2 + x_3 \le 100$

    $5x_1 + 6x_2 + 5x_3 \le 400$

    $x_1, x_2, x_3 \ge 0.$

$x_1 = 0,\ x_2 = 0, x_3 = 80$, Max $Z = 150000.$

14. Vertices of the feasible region are: $\left(\frac{1}{2},0\right),\left(0,\frac{1}{3}\right)$.

15. Vertices of the feasible region are: $(0,0),\left(\frac{7}{2},0\right),\left(\frac{7}{2},\frac{5}{3}\right),\left(\frac{8}{5},\frac{12}{5}\right),(0,3)$

$x_1 = 8/5,\ x_2 = 12/5,\quad \text{Max } Z = 124/5.$

Chapter III

9. $x_1 = 20/19,\ x_2 = 45/19,\ \text{Max } Z = 235/19.$

10. $x_1 = 3,\ x_2 = 0,\ \text{Max } Z = 21.$

11. Solution is unbounded.

12. Solution is unbounded.

13. $x_1 = 0,\ x_2 = 0,\ x_3 = 5,\ \text{Mini } Z = -5.$

14. $x_1 = 50/7,\ x_2 = 0,\ x_3 = 55/7,\ x_4 = 0,\ \text{Mini } Z = -695/7.$

Chapter IV

5. The problem is Max $Z = 60,000x_1 + 12,000x_2$

Subject to

$9,000x_1 + 12,000x_2 \le 7,20,000$

$x_1 \ge 2$

$x_2 \ge 3$

$x_1, x_2 \ge 0$.

Solution is $x_1 = 2,\ x_2 = 58.5$ Max $Z = 7,14,000.$

6. The problem is Max $Z = 50x_1 + 30x_2$

Subject to

$2x_1 + x_2 \ge 8$

$x_1 + x_2 \ge 12$

$3x_1 + 2x_2 \le 34$

$x_1, x_2 \ge 0.$

Solution is $x_1 = 10,\ x_2 = 2$ Max $Z = 560.$

7. The problem is Max $Z = 300x_1 + 400x_2$

Subject to

$5x_1 + 4x_2 \le 200$

$3x_1 + 5x_2 \le 150$

$5x_1 + 4x_2 \ge 100$

$8x_1 + 4x_2 \ge 80$

$x_1, x_2 \ge 0.$

Solution is $x_1 = 400/13,\ x_2 = 150/13$ Max $Z = 18000/13.$

8. The problem is Min $Z = 3x_1 + 8x_2$

Subject to

$x_1 + x_2 = 200$

$x_1 \le 80$

$x_2 \ge 60$

$x_1, x_2 \ge 0.$

Solution is $x_1 = 80,\ x_2 = 120$ Min $Z = 1200.$

9. The problem is Min $Z = 5000x_1 + 7000x_2$

Subject to

$100x_1 + 120x_2 \ge 5000$

$200x_1 + 120x_2 \ge 6000$

$200x_1 + 400x_2 \ge 14000$

$x_1, x_2 \ge 0.$

Solution is $x_1 = 20,\ x_2 = 25$ Min $Z = 2,75,000.$

10. $x_1 = 2,\ x_2 = 0$ Max $Z = 6.$

11. $x_1 = 0,\ x_2 = 5$ Max $Z = 40.$

12. $x_1 = 1.2,\ x_2 = 0, x_3 = 0.9,\ x_4 = 0,$ Max $Z = 19.8.$

13. Here all $\Delta_j \ge 0$, but at the same time artificial variable A_1 appears in the basis. Hence, the given LPP has no feasible solution.

14. No solution.

15. $x_1 = 23/3,\ x_2 = 5, x_3 = 0,$ Max $Z = 85/3.$

16. There does not exist any feasible solution.

17. $x_1 = 0,\ x_2 = 100, x_3 = 0,$ Max $Z = 140000.$

18. $x_1 = 100,\ x_2 = 100$ Max $Z = 60000.$

19. $x_1 = 55/7,\ x_2 = 30/7, x_3 = 0,$ Max $Z = 45/7.$

23. $x_4 = 25/3,\ x_5 = 16/3,\ x_6 = 0,$ Max $Z = 43/3.$

24. $x_1 = 5/2,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0,$ Min $Z = 10.$

25. Unbounded solution.

26. $x_1 = 4,\ x_2 = 1,$ Min $Z = 11.$

27. $x_1 = 11/3,\ x_2 = 0, x_3 = 14/3,$ Min $Z = 47/3.$

Chapter V

7. Minimize $Z^* = 10y_1 + 2y_2 + 6y_3$

   Subject to the constraints

   $$y_1 + 2y_2 + 2y_3 \geq 1$$
   $$y_1 + 0y_2 - 2y_3 \geq -1$$
   $$y_1 - y_2 - 3y_3 \geq 3$$
   $$y_1, y_2, y_3 \geq 0.$$

8. Maximize $Z^* = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

   Subject to the constraints

   $$3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$$
   $$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$$
   $$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$$
   $$y_1, y_2, y_3, y_4, y_5 \geq 0.$$

9. Maximize $Z^* = -160y_1 + 30y + 10y_4$

   Subject to the constraints

   $$-2y_1 + y + y_4 \leq 1$$
   $$-4y_1 - y \leq 2$$
   $$y_1, y_4 \geq 0$$

   where, $y = y_2 - y_3,\ y$ being unrestricted in sign.

10. Maximize $Z^* = -7y_1 + 12y_2 + 10y_3$

Subject to the constraints

$-3y_1 + 2y_2 - 4y_3 \le 1$

$y_1 - 4y_2 + 3y_3 \le -3$

$-2y_1 \quad + 8y_3 \le -2$

$y_1, y_2 \ge 0$, y_3 is unrestricted in sign.

11. Minimize $Z^* = 2y_1 + y_2$

Subject to the constraints

$-2y_1 + 2y_2 \ge 1$

$y_1 + 3y_2 \ge -2$

$3y_1 + 4y_2 \ge 3$

y_1, y_2 unrestricted in sign.

12. $x_1 = 9/2, x_2 = 3/2$, Max $z = 27/2$.

13. $x_1 = 5/19, x_2 = 16/19$, Min $z = 235/19$.

14. $x_1 = 4, x_2 = 3$, Max $z = 18$.

15. $x_1 = 21/4, x_2 = 0, x_3 = 0, x_4 = 0$, Max $z = 63/2$

16. Feasible solution does not exist.

17. $x_1 = 4$, $x_2 = 3$, Max $z = 25$.

18. Problem has no solution.

19. $x_1 = 1/4$, $x_2 = 5/4$, $x_3 = 0$, Min z = 10.

Chapter VI

16. Use VAM.

$x_{11} = 14, x_{15} = 6, x_{21} = 4, x_{23} = 8, x_{24} = 18, x_{31} = 15, x_{41} = 7, x_{42} = 6$

Minimum cost = Rs.321.

17. Use VAM. Alternative solution exists.

$x_{12} = 20, x_{13} = 3, x_{14} = 7, x_{21} = 10, x_{22} = 5,\ x_{33} = 15$

18. Use VAM.

$x_{11} = 30, x_{12} = 70, x_{13} = 50, x_{24} = 40, x_{31} = 60,\ x_{34} = 20$

Minimum cost is Rs.8,190.

19. Use VAM.

$x_{11} = 40, x_{12} = 26, x_{14} = 4, x_{24} = 38, x_{32} = 2,\ x_{33} = 30$

Minimum cost = Rs.86.

20. $x_{12} = 15, x_{13} = 85, x_{22} = 20, x_{24} = 105, x_{31} = 60, x_{32} = 45,\ x_{35} = 70$

Minimum cost = Rs.7,605.

21. i. $x_{11} = 1, x_{13} = 1, x_{21} = 7, x_{15} = 2, x_{32} = 3,\ x_{33} = 6, x_{44} = 2$ Min cost = Rs.56.

ii. $x_{12} = 5, x_{13} = 1, x_{24} = 1, x_{31} = 7, x_{33} = 2,\ x_{34} = 1$ Min cost = Rs.102.

Alternative solution exists.

22. $x_{14} = 300, x_{21} = 200, x_{22} = 300,\ x_{33} = 500, x_{34} = 100$ Min cost = Rs.15,500.

23. Total shipment time is 9 units. Details of plan are:

$x_{11} = 2$ with $t_{11} = 6$; $x_{13} = 3$ with $t_{13} = 3$; $x_{23} = 7$ with $t_{23} = 1$;
$x_{31} = 8$ with $t_{31} = 6$; and $x_{42} = 5$ with $t_{42} = 9$.

24.

		1	2	3^1	a_i
Regular time	1	1	1.50	0	920
(over time)	1^1	2.25	2.75	0	920
Regular time	2	∞	1	0	250
(over time)	2^1	∞	2.25	0	250
	b_j	800	1400	140	

Chapter VII

14. A $\rightarrow$ 1; B $\rightarrow$ 4; C $\rightarrow$ 2; D $\rightarrow$ 3. minimum cost = 78

Alternate solution exists.

15. $D_1 \rightarrow C_4$, $D_2 \rightarrow C_3$, $D_3 \rightarrow C_2$, $D_4 \rightarrow C_5$, $D_5 \rightarrow C_1$. Minimum distance = 450 km.

16. Operator 1 $\rightarrow$ job 4, Operator 2 $\rightarrow$ job 1, Operator 3 $\rightarrow$ job dummy (6), Operator 4 $\rightarrow$ job 5, Operator 5 $\rightarrow$ job 2, Operator 6 $\rightarrow$ job 3.

Total minimum completion of time is 16.

17. Dummy $\rightarrow$ Calcutta, Mr. X $\rightarrow$ Bombay, Mr. Y $\rightarrow$ Madras, Mr. Z $\rightarrow$ Delhi.

    The cost of relocation for this assignment is Rs. 48,000.

18. Job 1 $\rightarrow$ Machine 3, Job 2 $\rightarrow$ Machine 5, Job 3 $\rightarrow$ Machine 4,

    Job 4 $\rightarrow$ Machine 2, Job 5 $\rightarrow$ Machine 1. Min cost is 220.

19. Job 1 $\rightarrow$ Machine C, Job 2 $\rightarrow$ Machine E, Job 3 $\rightarrow$ Machine A,

    Job 4 $\rightarrow$ Machine D, Job 5 $\rightarrow$ Machine B. Min cost is 214.

20. A $\rightarrow$ 5, B $\rightarrow$ 2, C $\rightarrow$ 1, D $\rightarrow$ 3, E $\rightarrow$ 4, F $\rightarrow$ 6; Min cost is 185.

21. Truck 1 $\rightarrow$ Spot 9, Truck 2 $\rightarrow$ Spot 8, Truck 3 $\rightarrow$ Spot 7, Truck 4 $\rightarrow$ Spot 12, Spot 10 and Spot 11 are vacant. Min cost is 8.

22. $S_1 \rightarrow T_1$, $S_2 \rightarrow T_4$, $S_3 \rightarrow T_3$, $S_4 \rightarrow T_2$. Maximum profit 139.

23. A $\rightarrow$ E, E $\rightarrow$ C, C $\rightarrow$ B, B $\rightarrow$ D, D $\rightarrow$ A; Min cost is 21.

Glossary

Alternative Optimal Solution : It is a point in the feasible region that gives the same optimal value of the objective function.

Artificial Variable : A non-negative quantity added to the left side of a constraint of an LP problem in standard form.

Assignment Problem : A linear programming problem in which the aim is to allocate individuals to task in a manner that optimizes their overall effectiveness or cost.

Basic Feasible Solution : A feasible solution involving exactly *m* non-zero decision variables.

Basic Variable : A variable in the basis.

Basis : It is the vector consisting of basic variables.

Constraint : A constraint represents a mathematical equation regarding limitations imposed by the problem characteristics.

Decision Variable : Decision variables are the unknowns to be determined from the solution of an LPP model.

Degenerate Solution : A basic feasible solution involving one or more basic variables zero.

Duality : A property of an optimization problem. It relates any linear maximization problem to an equivalent minimization problem.

Feasible Solution : It is a set of values of the decision variables which satisfies all the constraints.

Hungarian Method : Method of solving the assignment problem.

Infeasible Solution : A solution not satisfying all the constraints.

Initial Basic Feasible Solution : It is the starting solution.

Linear Programming : Linear programming deals with problems in which the objective function, as well as constraints, are expressed as linear mathematical functions.

Objective Function : An objective function represents the mathematical equation of the major goal of the system in terms of unknowns called decision variables.

Operations Research : Operations Research concerned applying scientific methods to problems faced by executive and administrative authorities.

Optimal Solution : It is a feasible solution which optimizes the objective function.

Redundant Constraint : A constraint not satisfying the boundaries of feasible solution.

Slack Variable : It is the non-negative variable which is added to the left hand side of the constraint to convert it into equation.

Surplus Variable : It is the non-negative variable which is subtracted from the left hand side of the constraint to convert it into an equation.

Transportation Problem : A linear programming problem concerned with identifying optimum method for moving commodities from one set of locations to another with less cost.

Traveling Salesman Problem : A network problem that can be formulated as a combinatorial optimization of route of traveling by a salesman.

Unbounded Solution : An infinitely large solution.

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